Direct Encodings of NP-Complete Problems into Horn Sequents of Multiplicative Linear Logic

AIST

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Motivation

- To solve NP-complete problems
- Success of SAT solvers to solve NP-complete problems at a practical level
- Another Logical Viewpoint: Linear Logic
- Provability of Multiplicative Linear Logic (MLL) is NP-complete
- Any NP-complete problem can be encoded into MLL in principle
- No obvious existence of a direct encoding of a particular NP-complete problem

In this talk

- In the proceedings paper

1. Encodings of 3D MATCHING and PARTITION into MLL
2. Their correctness proofs using MLL proof nets

- In this talk

1. Encodings of these problems into HMLL
2. Only examples
3. Horn programs of these examples

The system IMLL

Formulas:

$$A ::= p \mid A \multimap B \mid A \otimes B$$

Inference rules:

I $\dfrac{}{A \vdash A}$

L $\multimap$ $\dfrac{\Sigma_1 \vdash A \qquad B, \Sigma_2 \vdash C}{\Sigma_1, A \multimap B, \Sigma_2 \vdash C}$

R $\multimap$ $\dfrac{\Sigma, A \vdash B}{\Sigma \vdash A \multimap B}$

L$\otimes$ $\dfrac{\Sigma, A, B \vdash C}{\Sigma, A \otimes B \vdash C}$

R$\otimes$ $\dfrac{\Sigma_1 \vdash A \qquad \Sigma_2 \vdash B}{\Sigma_1, \Sigma_2 \vdash A \otimes B}$

$\Sigma, \Sigma_1, \Sigma_2$ are multisets of IMLL formulas

Difference between IMLL and classical (or intuitionistic) logic

$$\dfrac{\dfrac{p \vdash p \qquad \dfrac{\dfrac{\dfrac{p \vdash p}{p, q \vdash p}\text{(W)} \qquad \dfrac{q \vdash q}{p, q \vdash q}\text{(W)}}{p, q \vdash p \wedge q} \qquad q \vdash q}{p, q, p \wedge q \to q \vdash q}}{p, p, p \to q, p \wedge q \to q \vdash q}}{p, p \to q, p \wedge q \to q \vdash q}\text{(C)}$$

Difference between IMLL and classical (or intuitionistic) logic (Cont.)

- But,

$$p, p \multimap q, p \otimes q \multimap q \vdash q$$

cannot be proved in IMLL

- No contraction and weakening rules in IMLL
- IMLL is more resource sensitive than classical (or intuitionistic) logic

The system HMLL

Simple Formulas:

$$X ::= p \mid X \otimes Y$$

Horn Implications:

$$X \multimap Y$$

Horn sequents:

$$X, \Gamma \vdash Y$$

where Γ is a multiset of Horn implications

The system HMLL (cont.)

Inference rules:

$$\text{I}\quad \frac{}{X \vdash X} \qquad \text{H}\quad \frac{}{X, (X \multimap Y) \vdash Y}$$

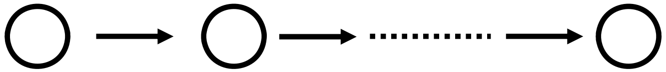
$$\text{S}\quad \frac{X_1 \otimes \cdots \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_n, \Gamma \vdash Z}{X_1 \otimes \cdots \otimes X_{i+1} \otimes X_i \otimes \cdots \otimes X_n, \Gamma \vdash Z}$$

$$\text{L}\otimes\quad \frac{X, \Gamma \vdash Y}{X \otimes V, \Gamma \vdash Y \otimes V} \qquad \text{Cut}\quad \frac{W, \Gamma \vdash U \quad U, \Gamma' \vdash Z}{W, \Gamma, \Gamma' \vdash Z}$$

HMLL is a very restricted subsystem of IMLL

Multiplicative Horn Programs

Directed chains:



vertices: simple formulas $X_1 \otimes \cdots \otimes X_n$

edges: Horn implications formulas $X \multimap Y$

such that

$$\begin{tikzcd} X_1 \otimes \cdots \otimes X_k \otimes X \otimes X_{k+1} \otimes \cdots \otimes X_n \arrow[d, "X \multimap Y"] \\ X_1 \otimes \cdots \otimes X_k \otimes Y \otimes X_{k+1} \otimes \cdots \otimes X_n \end{tikzcd}$$

$X_1 \otimes \cdots \otimes X_k \otimes X_{k+1} \otimes \cdots \otimes X_n$ and

$X_1 \otimes \cdots \otimes X_{k+1} \otimes X_k \otimes \cdots \otimes X_n$ are identified

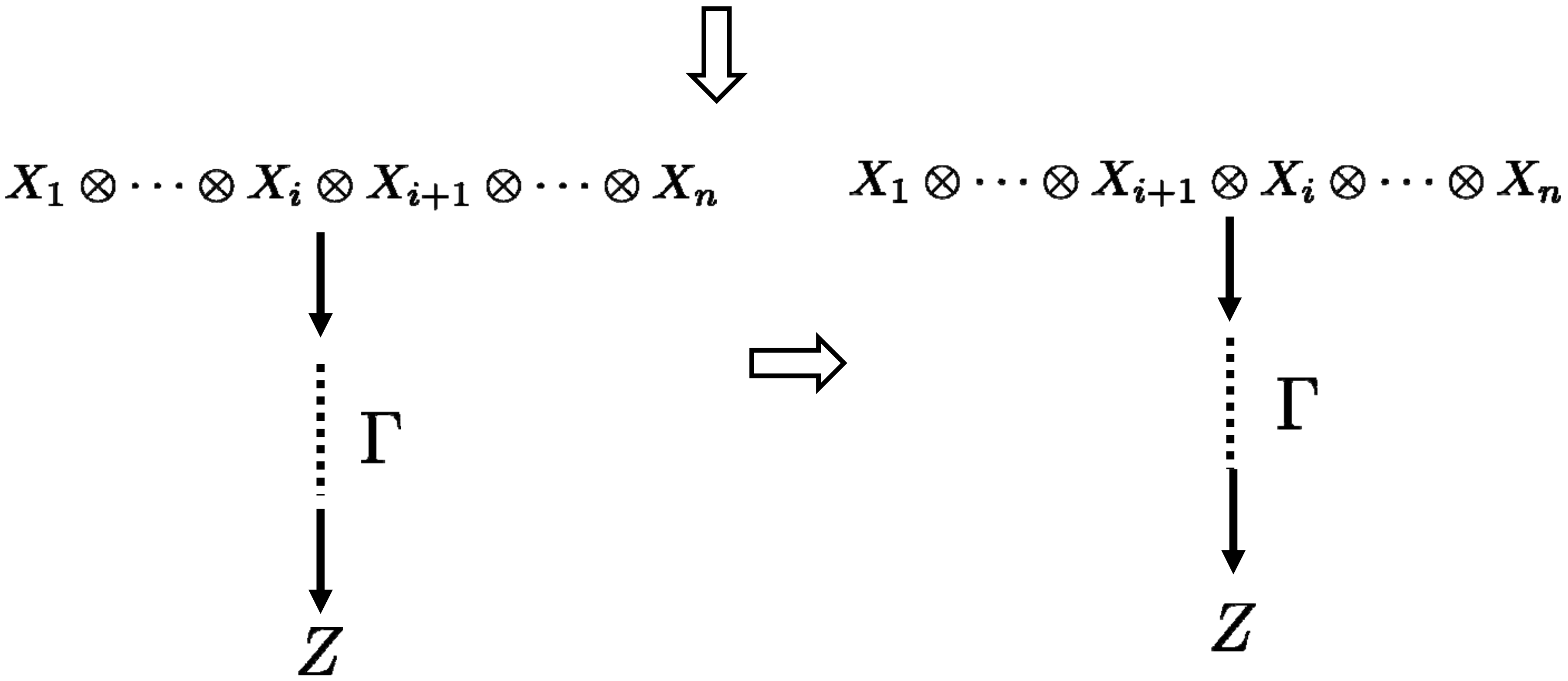
Interpretation of HMLL into Horn programs

$$\mathrm{I} \quad \frac{}{X \vdash X} \quad \Longrightarrow \quad X$$

$$\mathrm{H} \quad \frac{}{X, (X \multimap Y) \vdash Y} \quad \Longrightarrow \quad \begin{tikzcd} X \arrow[d, "X \multimap Y"] \\ Y \end{tikzcd}$$

Interpretation of HMLL into Horn programs (Cont.)

$$\text{S}\quad \frac{X_1 \otimes \cdots \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_n, \Gamma \vdash Z}{X_1 \otimes \cdots \otimes X_{i+1} \otimes X_i \otimes \cdots \otimes X_n, \Gamma \vdash Z}$$



Interpretation of HMLL into Horn programs (Cont.)

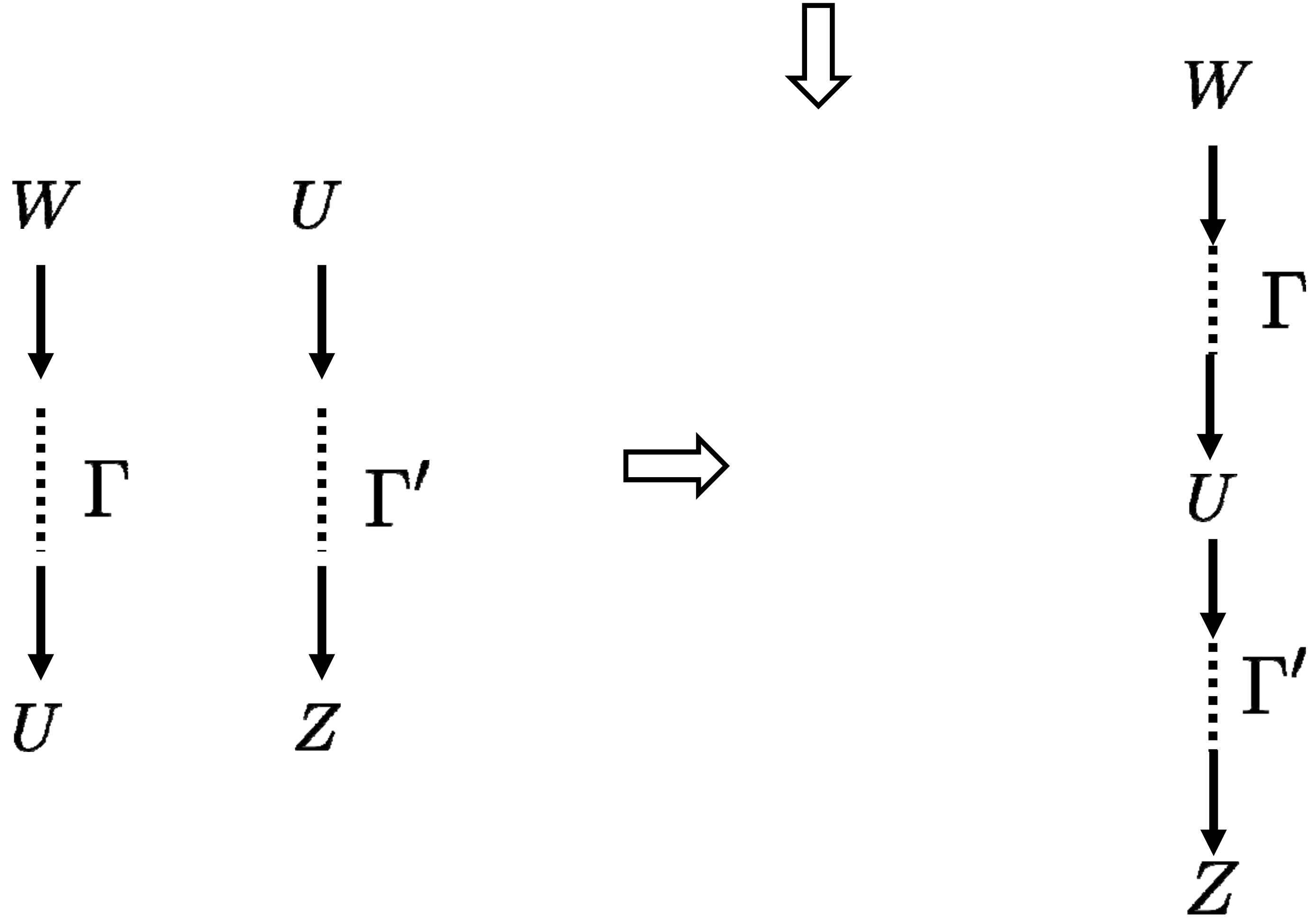
$$\mathrm{L}\otimes \quad \frac{X, \Gamma \vdash Y}{X \otimes V, \Gamma \vdash Y \otimes V}$$

$$\Downarrow$$

$$\begin{tikzcd}
X \arrow[d, dotted, "\Gamma"] & & X \otimes V \arrow[d, dotted, "\Gamma"] \\
Y \arrow[r, phantom, "\Rightarrow"] & & Y \otimes V
\end{tikzcd}$$

Interpretation of HMLL into Horn programs (Cont.)

$$\text{Cut}\quad \frac{W,\Gamma\vdash U \qquad U,\Gamma'\vdash Z}{W,\Gamma,\Gamma'\vdash Z}$$



Multiplicative Horn Programs

Theorem (Kanovich)

$\vdash X, \Gamma \vdash Z$ is provable in HMLL
iff there is a Multiplicative Horn program
from X to Y such that
each Horn implication in Γ occurs
as an edge exactly once.

The 3D MATCHING Problem

Given $T \subseteq A \times B \times C,$ where $|A| = |B| = |C| = n,$

Find $T_0 \subseteq T$ such that

$$\begin{array}{rcl} |T_0| & = & n \\ A & = & \{a \in A \,|\, \exists b \in B. \exists c \in C. \langle a, b, c\rangle \in T_0\} \\ B & = & \{b \in B \,|\, \exists a \in A. \exists c \in C. \langle a, b, c\rangle \in T_0\} \\ C & = & \{c \in C \,|\, \exists a \in A. \exists b \in b. \langle a, b, c\rangle \in T_0\} \end{array}$$

The 3D MATCHING Problem (Example)

Given

$$\begin{aligned} A &= \{a_1, a_2, a_3\} \\ B &= \{b_1, b_2, b_3\} \\ C &= \{c_1, c_2, c_3\} \\ T &= \{\langle a_1, b_1, c_2\rangle, \langle a_1, b_2, c_3\rangle, \langle a_2, b_2, c_1\rangle, \langle a_2, b_3, c_1\rangle, \langle a_3, b_1, c_2\rangle\} \end{aligned}$$

Find $T_0 \subseteq T$ such that

$$\begin{aligned} |T_0| &= 3 \\ A &= \{a \in A \,|\, \exists b \in B. \exists c \in C. \langle a, b, c\rangle \in T_0\} \\ B &= \{b \in B \,|\, \exists a \in A. \exists c \in C. \langle a, b, c\rangle \in T_0\} \\ C &= \{c \in C \,|\, \exists a \in A. \exists b \in b. \langle a, b, c\rangle \in T_0\} \end{aligned}$$

Solution:

$$T_0 = \{\langle a_1, b_2, c_3\rangle, \langle a_2, b_3, c_1\rangle, \langle a_3, b_1, c_2\rangle\}$$

The 3D MATCHING Problem (Example)

$$\begin{array}{rcll}
A_T & = & \{a_1, a_1, a_2, a_2, a_3\}_{\mathbf{mul}} & \text{(first coordinate multiset of } T\text{)} \\
B_T & = & \{b_1, b_2, b_2, b_3, b_1\}_{\mathbf{mul}} & \text{(second coordinate multiset of } T\text{)} \\
C_T & = & \{c_2, c_3, c_1, c_1, c_2\}_{\mathbf{mul}} & \text{(third coordinate multiset of } T\text{)} \\
A_{\mathbf{co}} & = & A_T - A = \{a_1, a_2\}_{\mathbf{mul}} & \\
B_{\mathbf{co}} & = & B_T - B = \{b_1, b_2\}_{\mathbf{mul}} & \\
C_{\mathbf{co}} & = & C_T - C = \{c_1, c_2\}_{\mathbf{mul}} &
\end{array}$$

$$\Gamma_{\mathtt{3D\,MATCHING}} =$$

$$(b_1 \otimes b_2) \otimes (c_1 \otimes c_2), \qquad \text{from } B_{\mathbf{co}}, C_{\mathbf{co}}$$

$$a_1 \otimes a_2 \multimap ((b_1 \otimes b_2 \otimes b_3) \otimes (c_1 \otimes c_2 \otimes c_3)), \quad \text{from } A_{\mathbf{co}}, B, C$$

$$\left.\begin{array}{l}
b_1 \otimes c_2 \multimap a_1 \\
b_2 \otimes c_3 \multimap a_1 \\
b_2 \otimes c_1 \multimap a_2 \\
b_3 \otimes c_1 \multimap a_2 \\
b_1 \otimes c_2 \multimap a_3
\end{array}\right\} \quad \text{from } T = \{\langle a_1, b_1, c_2\rangle, \langle a_1, b_2, c_3\rangle, \langle a_2, b_2, c_1\rangle, \langle a_2, b_3, c_1\rangle, \langle a_3, b_1, c_2\rangle\}$$

$$\vdash a_1 \otimes a_2 \otimes a_3 \qquad \text{from } A$$

The 3D MATCHING Problem (Example)

$$\begin{tikzcd}
(b_1 \otimes b_2) \otimes (c_1 \otimes c_2) =_{\mathrm{mul}} (b_1 \otimes c_2) \otimes (b_2 \otimes c_1) \arrow[d, "b_1 \otimes c_2 \multimap a_1"] \\
a_1 \otimes (b_2 \otimes c_1) \arrow[d, "b_2 \otimes c_1 \multimap a_2"] \\
a_1 \otimes a_2 \arrow[d, "a_1 \otimes a_2 \multimap ((b_1 \otimes b_2 \otimes b_3) \otimes (c_1 \otimes c_2 \otimes c_3))"] \\
(b_1 \otimes b_2 \otimes b_3) \otimes (c_1 \otimes c_2 \otimes c_3) \\
=_{\mathrm{mul}} (b_2 \otimes c_3) \otimes (b_3 \otimes c_1) \otimes (b_1 \otimes c_2)
\end{tikzcd}$$

The 3D MATCHING Problem (Example)

$$\begin{tikzcd}
{} \arrow[d, "b_2 \otimes c_3 \multimap a_1"] \\
a_1 \otimes (b_3 \otimes c_1) \otimes (b_1 \otimes c_2) \arrow[d, "b_3 \otimes c_1 \multimap a_2"] \\
a_1 \otimes a_2 \otimes (b_1 \otimes c_2) \arrow[d, "b_1 \otimes c_2 \multimap a_3"] \\
a_1 \otimes a_2 \otimes a_3
\end{tikzcd}$$

So, we have obtained a Horn program for the sequent $\Gamma_{\mathbf{3D\,MATCHING}}$

The PARTITION problem

Given a finite set A and a function $s : A \to \mathbb{Z}^+$

Find a subset $A' \subseteq A$ such that

$$\sum_{s \in A'} s(a) = \sum_{s \in A - A'} s(a)$$

Example: $A = \{a_1, a_2, a_3, a_4\}$

$$s = \{a_1 \mapsto 2, a_2 \mapsto 3, a_3 \mapsto 2, a_4 \mapsto 1\}$$

A solution: $A' = \{a_1, a_3\}$

$$\sum_{s \in A} s(a) = 8$$

The PARTITION problem

$$\Gamma_{\mathtt{PARTITION}} =$$

$$a_1 \otimes a_2 \otimes a_3 \otimes a_4, \quad \text{from} \quad A = \{a_1, a_2, a_3, a_4\}$$

$$\left.\begin{array}{l} a_1 \multimap b \otimes b, \\ a_2 \multimap b \otimes b \otimes b, \\ a_3 \multimap b \otimes b, \\ a_4 \multimap b, \\ a_1 \multimap c \otimes c, \\ a_2 \multimap c \otimes c \otimes c, \\ a_3 \multimap c \otimes c, \\ a_4 \multimap c, \end{array}\right\} \text{from } s = \{a_1 \mapsto 2, a_2 \mapsto 3, a_3 \mapsto 2, a_4 \mapsto 1\}$$

$$(b \otimes b \otimes b \otimes b) \otimes (c \otimes c \otimes c \otimes c) \multimap a_1 \otimes a_2 \otimes a_3 \otimes a_4,$$

$$(b \otimes b \otimes b \otimes b) \otimes \underbrace{(c \otimes c \otimes c \otimes c)} \multimap e$$

$$\vdash e$$

$$\frac{\sum_{s \in A} s(a)}{2} = 4$$

The PARTITION problem

$$\begin{tikzcd}
a_1 \otimes a_2 \otimes a_3 \otimes a_4 \arrow[d, "a_1 \multimap b \otimes b"] \\
(b \otimes b) \otimes a_2 \otimes a_3 \otimes a_4 \arrow[d, "a_1 \multimap c \otimes c \otimes c"] \\
(b \otimes b) \otimes (c \otimes c \otimes c) \otimes a_3 \otimes a_4 \arrow[d, "a_3 \multimap b \otimes b"] \\
(b \otimes b) \otimes (c \otimes c \otimes c) \otimes (b \otimes b) \otimes a_4 \arrow[d, "a_4 \multimap c"] \\
(b \otimes b) \otimes (c \otimes c \otimes c) \otimes (b \otimes b) \otimes c
\end{tikzcd}$$

$$=_{\mathrm{mul}} (b \otimes b \otimes b \otimes b) \otimes (c \otimes c \otimes c \otimes c)$$

The PARTITION problem

$$\Big\downarrow\ (b \otimes b \otimes b \otimes b) \otimes (c \otimes c \otimes c \otimes c) \multimap a_1 \otimes a_2 \otimes a_3 \otimes a_4$$

$$a_1 \otimes a_2 \otimes a_3 \otimes a_4$$

$$\Big\downarrow\ a_1 \multimap c \otimes c$$

$$(c \otimes c) \otimes a_2 \otimes a_3 \otimes a_4$$

$$\Big\downarrow\ a_1 \multimap b \otimes b \otimes b$$

$$(c \otimes c) \otimes (b \otimes b \otimes b) \otimes a_3 \otimes a_4$$

$$\Big\downarrow\ a_3 \multimap c \otimes c$$

$$(c \otimes c) \otimes (b \otimes b \otimes b) \otimes (c \otimes c) \otimes a_4$$

The PARTITION problem

$$\Big\downarrow\; a_4 \multimap b$$

$$(c \otimes c) \otimes (b \otimes b \otimes b) \otimes (c \otimes c) \otimes b$$

$$=_{\mathbf{mul}} (b \otimes b \otimes b \otimes b) \otimes (c \otimes c \otimes c \otimes c)$$

$$\Big\downarrow\; (b \otimes b \otimes b \otimes b) \otimes (c \otimes c \otimes c \otimes c) \multimap e$$

$$e$$

So, we have obtained a Horn program for the sequent $\Gamma_{\mathtt{PARTITION}}$

Summary

- Have obtained direct encodings of two NP-complete problems into Horn programs
- A lot of work should be done:
  - More encodings
  - First-order extensions
  - Implementations, etc.