# Pre-orders for Reasoning about Stability

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## ABSTRACT

Pre-orders between processes, like simulation, have played a central role in the verification and analysis of discrete-state systems. Logical characterization of such pre-orders have allowed one to verify the correctness of a system by analyzing an abstraction of the system. In this paper, we investigate whether this approach can be feasibly applied to reason about stability properties of a system.

Stability is an important property of systems that have a continuous component in their state space; it stipulates that when a system is started somewhere close to its ideal starting state, its behavior is close to its ideal, desired behavior. In [6], it was shown that stability with respect to equilibrium states is not preserved by bisimulation and hence additional continuity constraints were imposed on the bisimulation relation to ensure preservation of Lyapunov stability. We first show that stability of trajectories is not invariant even under the notion of bisimulation with continuity conditions introduced in [6]. We then present the notion of uniformly continuous simulations — namely, simulation with some additional uniform continuity conditions on the relation — that can be used to reason about stability of trajectories. Finally, we show that uniformly continuous simulations are widely prevalent, by recasting many classical results on proving stability of dynamical and hybrid systems as establishing the existence of a simple, obviously stable system that simulates the desired system through uniformly continuous simulations.

## Categories and Subject Descriptors

D.2.4 [**Software**]: Software Engineering—*Software/Program Verification*

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Theory, Verification

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Stability, Pre-orders, Bisimulations, Uniform Continuity, Verification

## 1. INTRODUCTION

Bisimulation [23] is the canonical congruence that is used to understand when two systems are intended to be equivalent. It is taken to be the finest behavioral congruence that one would like to impose, and correctness specifications are often invariant under bisimulation, i.e., if two systems are bisimilar then either both satisfy the specification or neither one does. Given the efficiency of computing bisimulation quotients, bisimulation is often the basis of minimizing transition systems [22]. Variants of bisimulation, such as simulation are often used to *abstract* a system, and construct a simpler system that ignores some of the details of the system that maybe irrelevant to the satisfaction of the specification. Simulation and abstraction form the basis of verifying infinite state systems [5, 1].

Stability is the most common fundamental requirement imposed on dynamical and hybrid systems. Hybrid systems [17] are those whose system states evolve continuously with real-time modelling physical processes, while making occasional discrete mode changes, to reflect steps taken by a discrete, digital controller, or operating environment. Such models arise particularly naturally when describing embedded and cyber-physical systems. In such systems, stability is not just a design goal, but is often the principal requirement, so much so that unstable systems are deemed "unusable". Intuitively, stability requires that when a system is started somewhere close to its ideal starting state, its subsequent behavior is close to its ideally desired behavior. For example, it would not be acceptable for the performance of a robot to crucially depend on its initial position being known to infinite accuracy; more precisely, given any ideal starting orientation there should be some (open) neighborhood of this orientation for which all trajectories that start in this neighborhood remain close, and furthermore, it should be possible to ensure that the trajectories are as close as desired by making the neighborhood sufficiently small.

However, stability is not bisimulation invariant. This was first observed by Cuijpers in [6]. The stability requirement suggests that continuity requirements must be imposed on the witnessing bisimulation (or simulation) relation. Cuijpers considered the problem of Lyapunov stability of an equilibrium state $x_*$, which informally requires that if the system is started close to $x_*$ then it stays close to $x_*$ at all times. He showed that if a system $T_1$ with equilibrium point $x_*$ is simulated by $T_2$ with equilibrium point $y_*$ by a relation $R$ that relates $x_*$ and $y_*$, is upper semi-continuous, and $R^{-1}$ is lower semi-continuous [18], and if $T_2$ is Lyapunov stable near $y_*$ then $T_1$ is Lyapunov stable near $x_*$.

Cuijpers' result, unfortunately, does not extend when one considers stronger notions of stability, like asymptotic stability, or the (Lyapunov or asymptotic) stability of trajectories [1]. To see this, consider a standard dynamical system[2] $D_1$ that has two state variables $x, y$ taking values in $\mathbb{R}$, with the set of initial states being $\{(0, y) \mid y \in \mathbb{R}_{\geq 0}\}$. The execution map of $D_1$ is the function $f((x, y), t)$ which prescribes the state at time $t$ provided the state at time 0 was $(x, y)$; specifically, here $f((0, y), t) = (t, y)$. Observe that such a system is stable with respect to the trajectory $\tau = [t \mapsto (t, 0)]_{t \in \mathbb{R}_{\geq 0}}$, as executions that start close to $(0, 0)$ remain close to $\tau$ at all times. Let us consider another dynamical system $D_2$ that has the same state space, and same initial states, but whose execution map is $g((0, y), t) = (t, y(1 + t))$. Observe that this system is not stable with respect to its trajectory $\tau$, because no matter how close an initial condition $(0, y_0)$ is to the origin, the resulting execution $g((0, y_0), t)$ will diverge from $\tau$. On the other hand, the relation $R = \{((x_1, y_1), (x_2, y_2)) \mid x_2 = x_1 \text{ and } y_2 = y_1(1 + x_1)\}$ is a bisimulation between the systems $D_1$ and $D_2$. Observe that $R$ is bi-continuous and hence $R$ is the kind of bisimulation considered by Cuijpers.

In this paper, we identify congruences and pre-orders that allow one to reason about stability of trajectories. Our main observation is that in this case, *uniform* continuity conditions must be imposed on simulation and bisimulation relations. Thus, we introduce the notions of *uniformly continuous bisimulation* and *uniformly continuous simulation* and show that stability (Lyapunov or asymptotic) of trajectories is invariant under the new notion of bisimulation. Moreover we show that uniformly continuous simulations yield the right notion of abstraction for stability — if $D_1$ is uniformly simulated by $D_2$ and $D_2$ is stable then $D_1$ is also stable — yielding a mechanism for reasoning about stability.

Having established that uniformly continuous simulations and bisimulations define the right semantics for stability, we ask whether they arise naturally in practice. To substantiate the usefulness claim of the new relations, we investigate a number of classical results in control theory and hybrid systems, and show that the new pre-orders are widely prevalent and form the basis of stability proofs. The Hartman-Grobman theorem [14, 15, 16] is an important result that says that the behavior of any dynamical system near a hyperbolic equilibrium point is topologically the same as the behavior of a linear system near the same equilibrium point. We observe that, in fact, the Hartman-Grobman theorem establishes that there is a uniformly continuous bisimulation between the dynamical system and its linearization.

[1]Stability near an equilibrium point is the special case of stability of a trajectory, as the only trajectory from the equilibrium point stays at the equilibrium point.

[2]A standard dynamical system, in this paper, refers to a hybrid system without any discrete transitions.

Next we look at various results for establishing stability of dynamical and hybrid systems. The most common method for establishing stability of dynamical systems is that of Lyapunov theory [20], which requires finding a (Lyapunov) function from the state space of the dynamical system to $\mathbb{R}$ that is positive definite, and decreases along every behavior of the dynamical system. We observe that a Lyapunov function constructs a dynamical system whose stability is simple to prove. Moreover, the properties of a Lyapunov function ensure that the system constructed by the Lyapunov function uniformly continuously simulates original dynamical system. Thus, the proof of Lyapunov's theorem can be seen as constructing a simpler system which uniformly continuously simulates the original system and proving the stability of this simpler system. We also consider a technique for establishing the stability of a hybrid system using multiple Lyapunov functions. Once again we demonstrate that the result can be recast as saying that the existence of multiple Lyapunov functions of certain kind imply that the dynamical system can be abstracted (via uniformly continuous simulations) into a system for which stability can be proved easily, and therefore conclude the stability of the original hybrid system.

### Related Work.

Pre-orders and bisimulations have been widely used in the analysis of hybrid systems. Bisimulation relations have been widely used in safety verification of hybrid systems, and are the main technical tool in proving decidability of several subclasses of hybrid systems including timed and o-minimal systems [2, 21, 4]. The notion of approximate simulations and bisimulations have been introduced and used for simplifying the continuous dynamics and reducing the state space of standard dynamical and hybrid systems [10, 11, 12, 24] for safety verification.

Pre-orders and bisimulations to reason about stability were first considered by Cuijpers [6]. The notion of bi-continuous bisimulation was introduced as the semantic basis for reasoning about the Lyapunov stability of a single or set of equilibrium points. However, as argued in the introduction and in Section 4.1, this notion is not sufficient to reason about stronger stability notions like asymptotic stability or the stability of trajectories. We introduce uniformity conditions to reason about such stronger notions.

Finally, modal and temporal logics [8, 7, 9] have been extended with topological operators to reason about robustness of controllers. However, they are expressively inadequate to reason about the notions of stability considered here.

## 2. PRELIMINARIES

### Notation.

Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the set of reals and non-negative reals, respectively. Let $\mathbb{R}_\infty$ denote the set $\mathbb{R}_{\geq 0} \cup \{\infty\}$, where $\infty$ denotes the largest element of $\mathbb{R}_\infty$, that is, $x < \infty$ for all $x \in \mathbb{R}_{\geq 0}$. Also, for all $x \in \mathbb{R}_\infty$, $x + \infty = \infty$. Let $\mathbb{N}$ denote the set of all natural numbers $\{0, 1, 2, \cdots\}$, and let $[n]$ denote the first $n$ natural numbers, that is, $[n] = \{0, 1, 2, \cdots, n-1\}$. Let *Int* denote the set of all closed intervals of the form $[0, T]$, where $T \in \mathbb{R}_{\geq 0}$, and the infinite interval $[0, \infty)$.

### *Functions and Relations.*

Given a function $F$, let $Dom(F)$ denote the domain of $F$. Given a function $F : A \to B$ and a set $A' \subseteq A$, $F(A')$ denotes the set $\{F(a) \,|\, a \in A'\}$. Given a binary relation $R \subseteq A \times B$, $R^{-1}$ denotes the set $\{(x, y) \,|\, (y, x) \in R\}$. For a binary relation $R$, we will interchangeably use "$(x, y) \in R$" and "$R(x, y)$" to denote that $(x, y) \in R$.

### *Sequences.*

A sequence $\sigma$ is a function whose domain is either $[n]$ for some $n \in \mathbb{N}$ or the set of natural numbers $\mathbb{N}$. Length of a sequence $\sigma$, denoted $|\sigma|$, is $n$ if $Dom(\sigma) = [n]$ or $\infty$ otherwise. Given a sequence $\sigma : \mathbb{N} \to \mathbb{R}$ and an element $r$ of $\mathbb{R}_\infty$ we use $\sum_{i=0}^{\infty} \sigma(i) = r$ to denote the standard limit condition $\lim_{N\to\infty} \sum_{i=0}^{N} \sigma(i) = r$.

### *Extended Metric Space.*

An *extended metric space* is a pair $(M, d)$ where $M$ is a set and $d : M \times M \to \mathbb{R}_\infty$ is a distance function such that for all $m_1$, $m_2$ and $m_3$,

1. (Identity of indiscernibles) $d(m_1, m_2) = 0$ if and only if $m_1 = m_2$.
2. (Symmetry) $d(m_1, m_2) = d(m_2, m_1)$.
3. (Triangle inequality) $d(m_1, m_3) \leq d(m_1, m_2) + d(m_2, m_3)$.

When the metric on $M$ is clear we will simply refer to $M$ as a metric space.

Let us fix an extended metric space $(M, d)$ for the rest of this section. We define an open ball of radius $\epsilon$ around a point $x$ to be the set of all points which are within a distance $\epsilon$ from $x$. Formally, an *open ball* is a set of the form $B_\epsilon(x) = \{y \in M \,|\, d(x, y) < \epsilon\}$. An *open set* is a subset of $M$ which is a union of open balls. Given a set $X \subseteq M$, a *neighborhood* of $X$ is an open set in $M$ which contains $X$. Given a subset $X$ of $M$, an $\epsilon$-neighborhood of $X$ is the set $B_\epsilon(X) = \bigcup_{x\in X} B_\epsilon(x)$. A subset $X$ of $M$ is *compact* if for every collection of open sets $\{U_\alpha\}_{\alpha\in A}$ such that $X \subseteq \bigcup_{\alpha\in A} U_\alpha$, there is a finite subset $J$ of $A$ such that $X \subseteq \bigcup_{i\in J} U_i$.

### *Set Valued Functions.*

We consider set valued functions and define continuity of these functions. We choose not to treat set valued functions as single valued functions whose co-domain is a power set, since as argued in [18], it leads to strong notions of continuity, which are not satisfied by many functions. A *set valued function* $F : A \rightsquigarrow B$ is a function which maps every element of $A$ to a set of elements in $B$. Given a set $A' \subseteq A$, $F(A')$ will denote the set $\bigcup_{a\in A'} F(a)$. Given a binary relation $R \subseteq A \times B$, we use $R$ also to denote the set valued function $R : A \rightsquigarrow B$ given by $R(x) = \{y \,|\, (x, y) \in R\}$. Further, $F^{-1} : B \rightsquigarrow A$ will denote the set valued function which maps $b \in B$ to the set $\{a \in A \,|\, b \in F(a)\}$.

### *Continuity of Set Valued Functions.*

Let $F : A \rightsquigarrow B$ be a set valued function, where $A$ and $B$ are extended metric spaces. We define upper semi-continuity of $F$ which is a generalization of the "$\delta, \epsilon$ - definition" of continuity for single valued functions [18]. The function $F : A \rightsquigarrow B$ is said to be *upper semi-continuous* at $a \in Dom(F)$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } F(B_\delta(a)) \subseteq B_\epsilon(F(a)).$$

If $F$ is upper semi-continuous at every $a \in Dom(F)$ we simply say that $F$ is upper semi-continuous. Next we define a "uniform" version of the above definition, where, analogous to the case of single valued functions, corresponding to an $\epsilon$, there exists a $\delta$ which works for every point in the domain.

*Definition 1.* A function $F : A \rightsquigarrow B$ is said to be *uniformly continuous* if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall a \in Dom(A), F(B_\delta(a)) \subseteq B_\epsilon(F(a)).$$

We refer to uniform upper semi-continuity as just uniform continuity, because it turns out that the two notions of upper and lower semi-continuity coincide with the addition of uniformity condition, i.e., uniform upper semi-continuity is equivalent to uniform lower semi-continuity.

Next, we state some properties about upper semi-continuous and uniformly continuous functions.

PROPOSITION 1. *Let $F : A \rightsquigarrow B$ be a set-valued upper semi-continuous function. Then:*

- *$F^{-1}$ is also an upper semi-continuous function.*
- *If $A$ is compact, then $F$ is also uniformly continuous.*

## 3. HYBRID SYSTEMS

In this section, we present certain definitions related to hybrid systems. Hybrid systems are systems with mixed discrete-continuous behaviors, which are widely prevalent in various application domains including automotive, aeronautics, manufacturing and so on. There are many models for such systems, including the popular model of a hybrid automaton [17], which captures the discrete dynamics as a finite state automaton and the continuous dynamics as differential equations. In this exposition, we will not concern ourselves with any particular representation of these systems, but will use a generic semantic model with trajectories modeling continuous evolution and transitions modeling discrete transitions.

### 3.1 Hybrid Transition Systems

We begin by defining the two components of a hybrid transition system, namely, trajectories and transitions.

Given a set $S$, a *trajectory* over $S$ is a function $\tau : D \to S$, where $D \in Int$ is an interval. Let $Traj(S)$ denote the set of all trajectories over $S$. A *transition* over a set $S$ is a pair $\alpha = (s_1, s_2) \in S \times S$. Let $Trans(S)$ denote the set of all transitions over $S$.

*Definition 2.* A *hybrid transition system* ($HTS$) $\mathcal{H}$ is a tuple $(S, \Sigma, \Delta)$, where $S$ is a set of states, $\Sigma \subseteq Trans(S)$ is a set of transitions and $\Delta \subseteq Traj(S)$ is a set of trajectories.

**Notation** We will denote the elements of a $HTS$ using appropriate annotations, for example, the elements of $\mathcal{H}_i$ are $(S_i, \Sigma_i, \Delta_i)$, the elements of $\mathcal{H}'$ are $(S', \Sigma', \Delta')$ and so on.

Next, we define an execution of a hybrid transition system. We will need the notions of first and last elements of transitions and trajectories. For a trajectory $\tau$, $First(\tau) = \tau(0)$, and $Last(\tau)$ is defined only if $Dom(\tau)$ is a finite interval, and $Last(\tau) = \tau(T)$ where $Dom(\tau) = [0, T]$. For a transition $\alpha = (s_1, s_2)$, $First(\alpha) = s_1$ and $Last(\alpha) = s_2$. An execution is a finite or infinite sequence of trajectories and transitions which have matching end-points.

*Definition 3.* An *execution* of $\mathcal{H}$ is a sequence $\sigma : D \to \Sigma \cup \Delta$, where $D = [n]$ for some $n \in \mathbb{N}$ or $D = \mathbb{N}$, such that for each $0 \leq i < |\sigma| - 1$, $Last(\sigma(i)) = First(\sigma(i+1))$. Let $Exec(\mathcal{H})$ denote the set of all executions of $\mathcal{H}$.

In particular, this implies that all trajectories in an execution, except possibly the last, have finite domain.

In order to define distance between executions, we interpret an execution as a set which we call the graph of the execution. A graph of an execution consists of triples $(t, i, x)$ such that $x$ is a state that is reached after time $t$ has elapsed along the execution, and $i$ is the number of discrete transitions that have taken place before time $t$. Let us first define a function $Size : Traj(S) \cup Trans(S) \to \mathbb{R}_{\geq 0}$ which assigns a size to the trajectories and transitions. For $\tau \in Traj(S)$, $Size(\tau) = T$ if $Dom(\tau) = [0, T]$ and $Size(\tau) = \infty$ if $Dom(\tau) = [0, \infty)$. For $\alpha \in Trans(S)$, $Size(\alpha) = 0$.

*Definition 4.* For an execution $\sigma$ and $j \in Dom(\sigma)$, let $T_j = \sum_{k=0}^{j-1} Size(\sigma(k))$ and $K_j = |\{k \mid k < j,\ \sigma(k) \text{ is a transition}\}|$. The *graph* of an execution $\sigma$, denoted, $Gph(\sigma)$, is the set of all triples $(i, t, x)$ such that there exists $j \in Dom(\sigma)$ satisfying the following:

- $t \in [T_j, T_j + Size(\sigma(j))]]$.
- If $\sigma(j)$ is a trajectory, then $i = K_j$ and $x = \sigma(j)(t - T_j)$.
- If $\sigma(j)$ is a transition, then either $i = K_j$ and $x = First(\sigma)$, or $i = K_j + 1$ and $x = Last(\sigma)$.

Given a set of executions $\mathcal{T}$, we denote by $First(\mathcal{T})$ the set of starting points of executions in $\mathcal{T}$, that is, $First(\mathcal{T}) = \{First(\sigma(0)) \mid \sigma \in \mathcal{T}\}$. We will denote the set of states appearing in an execution $\sigma$ as $States(\sigma)$. For a transition $\alpha$, $States(\alpha) = \{First(\alpha), Last(\alpha)\}$, for a trajectory $\tau \in \Delta$, $States(\tau) = \{\tau(t) \mid t \in Dom(\tau)\}$, and for an execution $\sigma$, $States(\sigma) = \bigcup_{i \in Dom(\sigma)} States(\sigma(i))$.

Let $\mathcal{H} = (S, \Sigma, \Delta)$ be a hybrid transition system and $g : S \rightsquigarrow S'$ be a set valued function whose domain is the state space of $\mathcal{H}$. We extend $g$ to be a set valued function from $Traj(S)$ to $Traj(S')$ and from $Trans(S)$ to $Trans(S')$ as follows. Given a trajectory $\tau \in Traj(S)$, $g(\tau)$ is the set of trajectories $\tau'$ such that domain $Dom(\tau') = Dom(\tau)$ and $\tau'(t) \in g(\tau(t))$ for all $t \in Dom(\tau)$. Similarly, for a transition $\alpha = (s_1, s_2) \in Trans(S)$, $g(\alpha) = \{(s'_1, s'_2) \mid s'_1 \in g(s_1), s'_2 \in g(s_2)\}$. Also, for an execution $\sigma$ of $\mathcal{H}$, $g(\sigma)$ is the set of all $\sigma'$ such that $Dom(\sigma') = Dom(\sigma)$ and for each $i \in Dom(\sigma)$, $\sigma'(i) \in g(\sigma(i))$. If $g$ is a single valued function, then we use $g(\tau)$, $g(\alpha)$ and $g(\sigma)$ to denote the unique element mapped by $g$. We define $g(\mathcal{H})$ to be the *HTS* obtained by applying $g$ component-wise, that is, $g(\mathcal{H}) = (g(S), g(\Sigma), g(\Delta))$.

### *Metric Hybrid Transition Systems.*

A metric hybrid transition system is a hybrid transition system whose set of states is equipped with a metric. A *metric hybrid transition system* (*MHS*) is a pair $(\mathcal{H}, d)$ where $\mathcal{H} = (S, \Sigma, \Delta)$ is a hybrid transition system, and $(S, d)$ is an extended metric space. The metric $d$ on the state space can be lifted to executions, which will then be used to define stability. Before defining this extension, recall that given an extended metric space $(M, d)$, the *Hausdorff distance* between $A, B \subseteq M$, also denoted $d(A, B)$, is given by the maximum of

$$\{\sup_{p \in A} \inf_{q \in B} d(p, q), \sup_{p \in B} \inf_{q \in A} d(p, q)\}.$$

*Definition 5.* Let $(\mathcal{H}, d)$ be a metric transition system with $\mathcal{H} = (S, \Sigma, \Delta)$. For $(t_1, i_1, x_1), (t_2, i_2, x_2) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \times S$, let

$$d((t_1, i_1, x_1), (t_2, i_2, x_2)) = \max\{|t_1 - t_2|, |i_1 - i_2|, d(x_1, x_2)\}.$$

The *distance between executions* $\sigma_1, \sigma_2 \in Exec(\mathcal{H})$, denoted as $d(\sigma_1, \sigma_2)$, is defined as $d(Gph(\sigma_1), Gph(\sigma_2))$.

The above definition of distance between two hybrid executions is borrowed from [13].

Two executions are said to converge, if the distance between the two decreases as we consider smaller and smaller suffixes. Given a subset $G$ of $\mathbb{R}_{\geq 0} \times \mathbb{N} \times S$ and a $T \in \mathbb{R}_{\geq 0}$, let us denote by $G|_T$ the set $\{(t, i, x) \in G \mid t \geq T\}$.

*Definition 6.* Two executions $\sigma_1$ and $\sigma_2$ are said to *converge* if for every real $\epsilon > 0$, there exists a time $T \in \mathbb{R}_{\geq 0}$ such that $d(Gph(\sigma_1)|_T, Gph(\sigma_2)|_T) < \epsilon$.

We will use the predicate $Conv(\sigma_1, \sigma_2)$ to denote the fact that $\sigma_1$ and $\sigma_2$ converge.

## 3.2 Simulations and Bisimulations

We define the notion of simulation and bisimulation between hybrid transition systems along the lines of [19].

*Definition 7.* Given two hybrid transition systems $\mathcal{H}_1 = (S_1, \Sigma_1, \Delta_1)$ and $\mathcal{H}_2 = (S_2, \Sigma_2, \Delta_2)$, a binary relation $R \subseteq S_1 \times S_2$ is said to be a *simulation relation* from $\mathcal{H}_1$ to $\mathcal{H}_2$, denoted $\mathcal{H}_1 \preceq_R \mathcal{H}_2$, if for every $(s_1, s_2) \in R$, the following conditions hold:

- for every state $s'_1$ such that $(s_1, s'_1) \in \Sigma_1$, there exists a state $s'_2$ such that $(s_2, s'_2) \in \Sigma_2$ and $(s'_1, s'_2) \in R$; and
- for every trajectory $\tau_1 \in \Delta_1$ such that $First(\tau_1) = s_1$, there exists a trajectory $\tau_2 \in \Delta_2$ such that $First(\tau_2) = s_2$, and $\tau_2 \in R(\tau_1)$.

Intuitively, if there exists a simulation relation from $\mathcal{H}_1$ to $\mathcal{H}_2$, then $\mathcal{H}_2$ has more behaviors than $\mathcal{H}_1$. $\mathcal{H}_2$ is also referred to as an abstraction of $\mathcal{H}_1$. Simulations preserve various discrete time properties, such as, safety properties, in that, if $\mathcal{H}_1 \preceq_R \mathcal{H}_2$ and $\mathcal{H}_2$ satisfies the property, then we can conclude that $\mathcal{H}_1$ satisfies the property as well.

*Definition 8.* A binary relation $R \subseteq S_1 \times S_2$ is a *bisimulation relation* between $\mathcal{H}_1$ and $\mathcal{H}_2$, if $R$ is a simulation relation from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $R^{-1}$ is a simulation relation from $\mathcal{H}_2$ to $\mathcal{H}_1$.

We will use $\mathcal{H}_1 \sim_R \mathcal{H}_2$ to denote the fact that $R$ is a bisimulation relation between $\mathcal{H}_1$ and $\mathcal{H}_2$. So a bisimulation relation preserves properties in both directions, in that, $\mathcal{H}_1$ satisfies a bisimulation invariant property iff $\mathcal{H}_2$ satisfies it.

## 3.3 Stability of Hybrid Transition Systems

In this section, we introduce various properties related to the stability of systems (a good introductory book is [20]). Intuitively, stability is a property that requires that a system when started close to the ideal starting state, behaves in a manner that is close to its ideal, desired behavior.

### *Lyapunov Stability.*

We first define the notion of Lyapunov stability. Given a *HTS* $\mathcal{H}$ and a set of executions $\mathcal{T} \subseteq \mathit{Exec}(\mathcal{H})$, we say that $\mathcal{H}$ is *Lyapunov stable* (*LS*) with respect to $\mathcal{T}$, if for every $\epsilon > 0$ in $\mathbb{R}_{\geq 0}$, there exists a $\delta > 0$ in $\mathbb{R}_{\geq 0}$ such that the following condition holds:

$$\forall \sigma \in \mathit{Exec}(\mathcal{H}), d(\mathit{First}(\sigma(0)), \mathit{First}(\mathcal{T})) < \delta \implies \exists \rho \in \mathcal{T}, d(\sigma, \rho) < \epsilon. \quad (1)$$

The above statement says that for every execution $\sigma$ of the system $\mathcal{H}$ which starts with in a distance $\delta$ of some execution $\rho'$ in $\mathcal{T}$, there exists an execution $\rho$ in $\mathcal{T}$ which is with in distance $\epsilon$ from $\sigma$.

### *Asymptotic Stability.*

Next we define a stronger notion of stability called asymptotic stability which in addition to Lyapunov stability requires that the executions starting close also converge as time goes to infinity. A *HTS* $\mathcal{H}$ is said to be *asymptotically stable* (*AS*) with respect to a set of execution $\mathcal{T} \subseteq \mathit{Exec}(\mathcal{H})$, if it is Lyapunov stable and there exists a $\delta > 0$ in $\mathbb{R}_{\geq 0}$ such that

$$\forall \sigma \in \mathit{Exec}(\mathcal{H}), d(\mathit{First}(\sigma(0)), \mathit{First}(\mathcal{T})) < \delta \implies \exists \rho \in \mathcal{T}, \mathit{Conv}(\sigma, \rho). \quad (2)$$

So a system $\mathcal{H}$ is asymptotically stable with respect to a set of its executions $\mathcal{T}$ if $\mathcal{H}$ is Lyapunov stable with respect to $\mathcal{T}$ and every execution starting within a distance of $\delta$ from the starting point of some execution in $\mathcal{T}$ converges to some execution in $\mathcal{T}$.

*Remark 1.* The notions of stability with respect to an equilibrium point are a special case of the notion of stability with respect to trajectories, as an equilibrium point has the property that the only trajectory from the equilibrium point is one that stays there.

# 4. UNIFORMLY CONTINUOUS RELATIONS AND STABILITY PRESERVATION

The main focus of this paper is to examine the right preorders required to reason about stability properties. In the discrete setting, most interesting properties are known to be invariant under the classical notion of bisimulation. However, as shown in [6], stability is not invariant under bisimulation. That is, even with respect to a set of points (the trajectories in $\mathcal{T}$ essentially correspond to the trivial evolution of the equilibrium points), there are systems which are bisimilar, but such that only one of them is stable. Cuijpers [6] introduces bisimulations with additional continuity conditions and shows that they preserve stability with respect to a set of equilibrium points. More precisely, Cuijpers' result is as follows. Recall, a set of points $X$ is stable if for every open neighborhood $U$ of $X$, there exists a neighborhood $V$ of $X$ such that all trajectories starting from $V$ remain with in $U$. It is shown that if $R$ is a simulation with certain continuity conditions on $R$ and $R^{-1}$, then stability with respect to a set of points is preserved (See Theorem 2 of [6]). We observe that the notion of continuous bisimulation introduced in [6] does not suffice when one considers stability of trajectories. In fact, it does not even suffice to reason about asymptotic stability with respect to a set of points. Next, we discuss these observations; some of the details have been postponed to Appendix A.

## 4.1 Insufficiency of Continuity

### *Lyapunov Stability of Trajectories.*

Let us consider the dynamical systems $D_1$ and $D_2$ from the introduction. Note that system $D_1$ is Lyapunov stable with respect to the trajectory $[t \mapsto (t, 0)]_{t \in \mathbb{R}_{\geq 0}}$, and the system $D_2$ is not Lyapunov stable with respect to the same trajectory. However, the relation $R$ between the states of $D_1$ and $D_2$ is a bisimulation relation. Moreover, $R$ is bi-continuous, that is, both $R$ and $R^{-1}$ (when interpreted as single valued functions) are continuous. This shows that bisimulation even with additional continuity restrictions, which subsume the continuity restrictions in [6], does not suffice to preserve Lyapunov stability.

### *Asymptotic Stability of Trajectories.*

Next let us consider a dynamical system $D_3$ which is similar to $D_2$ except that $g((0, y), t) = ye^{-t}$. Note that $D_3$ is asymptotically stable. Then the relation $R'$ between $D_1$ and $D_3$ given by $\{((x_1, y_1), (x_2, y_2)) \mid x_1 = x_2 \text{ and } y_2 = y_1 e^{-x_1}\}$ is a bi-continuous bisimulation between $D_1$ and $D_3$, however, $D_3$ is asymptotically stable, where as $D_1$ is not. So the continuity conditions in [6] on bisimulation relations, do not suffice to reason about asymptotic stability of trajectories. In fact, they do not suffice even to reason about asymptotic stability with respect to a set of points (see Appendix A for more details).

## 4.2 Uniformly Continuous Simulations and Bisimulations

In this section, we introduce the notion of uniformly continuous simulations which add certain uniformity conditions on the relation, and show that they suffice to preserve both Lyapunov and asymptotic stability of trajectories.

*Definition 9.* A *uniformly continuous simulation* from a *HTS* $\mathcal{H}_1$ to a *HTS* $\mathcal{H}_2$ is a binary relation $R \subseteq S_1 \times S_2$ such that $R$ is a simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and $R$ and $R^{-1}$ are uniformly continuous functions.

The main result of this section is that uniformly continuous simulations serve as the right foundation for abstractions when verifying stability properties. That is, we will show that if $\mathcal{H}_1$ is uniformly continuously simulated by $\mathcal{H}_2$ and $\mathcal{H}_2$ is stable with respect to $\mathcal{T}_2$ then $\mathcal{H}_1$ will be stable with respect to $\mathcal{T}_1$. However, for such an observation to hold, the simulation relation between $H_1$ and $H_2$ should also relate the executions $\mathcal{T}_1$ and $\mathcal{T}_2$. So before proving the main result of this section, we first formally define how the simulation relation should relate the sets $\mathcal{T}_1$ and $\mathcal{T}_2$.

*Definition 10.* Given *HTS*s $\mathcal{H}_1$ and $\mathcal{H}_2$, and sets of executions $\mathcal{T}_1 \subseteq \mathit{Exec}(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq \mathit{Exec}(\mathcal{H}_2)$, a binary relation $R \subseteq S_1 \times S_2$ is said to be *semi-complete* with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ if the following hold:

- $R(\mathit{First}(\mathcal{T}_1)) = \mathit{First}(\mathcal{T}_2)$.
- For every $\rho_2 \in \mathcal{T}_2$, there is an execution in $\rho_1 \in \mathcal{T}_1$ such that $\rho_2 \in R(\rho_1)$.

- For every $x \in States(\mathcal{T}_2)$, $R^{-1}(x)$ is a singleton.
- There exists $\delta > 0$ such that for all $x \in B_\delta(First(\mathcal{T}_1))$, there exists a $y$ such that $R(x, y)$.

$R$ is *complete* with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ if $R$ and $R^{-1}$ are semi-complete with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$.

The next theorem states that uniformly continuous simulations preserve Lyapunov and asymptotic stability.

Theorem 1 (Stability Preservation Theorem). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hybrid transition systems and $\mathcal{T}_1 \subseteq Exec(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq Exec(\mathcal{H}_2)$ be two sets of execution. Let $R \subseteq S_1 \times S_2$ be a uniformly continuous simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let $R$ be semi-complete with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then the following hold:*

1. *If $\mathcal{H}_2$ is Lyapunov stable with respect to $\mathcal{T}_2$ then $\mathcal{H}_1$ is Lyapunov stable with respect to $\mathcal{T}_1$.*
2. *If $\mathcal{H}_2$ is asymptotically stable with respect to $\mathcal{T}_2$ then $\mathcal{H}_1$ is asymptotically stable with respect to $\mathcal{T}_1$.*

Proof. (Lyapunov stability preservation) Let $\mathcal{H}_2$ be Lyapunov stable with respect to $\mathcal{T}_2$. We will show that $\mathcal{H}_1$ is Lyapunov stable with respect to $\mathcal{T}_1$. Let us fix an $\epsilon > 0$. We need to show that there exists a $\delta > 0$ such that Equation (1) holds. The uniform continuity of $R^{-1}$ gives us an element of $\mathbb{R}_{\geq 0}$ corresponding to the $\epsilon$ above. Let us call it $\epsilon'$. We can assume that $\epsilon' < \epsilon$. Lyapunov stability of $\mathcal{T}_2$ gives us an element of $\mathbb{R}_{\geq 0}$ corresponding to the $\epsilon'$. Let us call it $\delta'$. Finally, uniform continuity of $R$ gives us an element of $\mathbb{R}_{\geq 0}$ corresponding to $\delta'$, which we call $\delta$. Let us assume without loss of generality that $\delta$ satisfies the last condition in the definition of semi-completeness of $R$ with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$.

Let $\sigma_1 \in Exec(\mathcal{H}_1)$ be such $d(First(\sigma_1(0)), First(\mathcal{T}_1)) < \delta$. We need to show that there exists a $\rho_1 \in \mathcal{T}_1$ such that $d(\sigma_1, \rho_1) < \epsilon$. $d(First(\sigma_1(0)), First(\mathcal{T}_1)) < \delta$ implies that for every $s \in R(First(\sigma_1(0)))$, there exists $s'$ in $R(First(\mathcal{T}_1))$ such that $d(s, s') < \delta'$ (uniform continuity of $R$). Consider a $\sigma_2$ in $Exec(\mathcal{H}_2)$ which simulates $\sigma_1$, that is, $\sigma_2 \in R(\sigma_1)$. Then since $First(\sigma_2)$ is in $R(First(\sigma_1(0)))$ (definition of simulation), there exists $s' \in R(First(\mathcal{T}_1))$ such that $d(s, s') < \delta'$, that is, $d(First(\sigma_2), First(\mathcal{T}_2)) < \delta'$ (since due to semi-completeness $R(First(\mathcal{T}_1)) = First(\mathcal{T}_2)$). Then from the Lyapunov stability of $\mathcal{H}_2$, there exists a $\rho_2$ in $\mathcal{T}_2$ such that $d(\sigma_2, \rho_2) < \epsilon'$. Let $\rho_1$ be a trajectory in $\mathcal{T}_1$ such that $\rho_2 \in R(\rho_1)$ ($\rho_1$ exists due to the second condition in the definition of semi-completeness). We will show that $d(\sigma_1, \rho_1) < \epsilon$.

Let $(t_1, i_1, x_1)$ be in $Gph(\sigma_1)$. Then $(t_1, i_1, y_1)$ is in $Gph(\sigma_2)$ for some $y_1 \in R(x_1)$. $(t_1, i_1, y_1)$ is with in distance $\epsilon'$ from some $(t_2, i_2, y_2)$ in $Gph(\rho_2)$. In particular, $d(y_1, y_2) < \epsilon'$, $|t_1 - t_2| < \epsilon'$ and $|i - j| < \epsilon'$. Since $R^{-1}(y_2)$ is a singleton (from the third condition of the definition of semi-completeness), say $x_2$, every point in $R^{-1}(y_1)$ is with in distance $\epsilon$ from $x_2$. In particular, $d(x_1, x_2) < \epsilon$, and hence $(t_1, i_1, x_1)$ is with in distance $\epsilon$ from $(t_2, i_2, x_2)$ (since $\epsilon' < \epsilon$). The argument is similar when we fix a $(t_2, i_2, x_2)$ in $Gph(\rho_1)$. Hence, $d(\sigma_1, \rho_1) < \epsilon$.

(Asymptotic stability preservation) Let $\mathcal{H}_2$ be asymptotically stable with respect to $\mathcal{T}_2$. We will show that $\mathcal{H}_1$ is asymptotically stable with respect to $\mathcal{T}_1$. Let $\delta' > 0$ be such that for every $\sigma_2$ starting with in a $\delta'$ ball of $First(\mathcal{T}_2)$, there exists a $\rho_2$ in $\mathcal{T}_2$, such that $Conv(\sigma_2, \rho_2)$. Let $\delta$ be an element of $\mathbb{R}_{\geq 0}$ given by the uniform continuity of $R$. We will show that for every $\sigma_1$ starting with in a $\delta$ ball of $First(\mathcal{T}_1)$, there exists a $\rho_1$ in $\mathcal{T}_1$, such that $Conv(\sigma_1, \rho_1)$. (This is enough since the preservation of Lyapunov stability follows from the previous part). Let us fix such a $\sigma_1$. Then similar to an argument in the previous part, there exists a $\sigma_2$ which simulates $\sigma_1$ and $d(First(\sigma_2), First(\mathcal{T}_2)) < \delta'$. Then from the asymptotic stability of $\mathcal{H}_2$ there exists $\rho_2$ in $\mathcal{T}_2$ such that $Conv(\sigma_2, \rho_2)$. Let $\rho_1$ be a trajectory in $\mathcal{T}_1$ such that $\rho_2 \in R(\rho_1)$. We will show that $Conv(\sigma_1, \rho_1)$. Let us fix an $\epsilon > 0$. We need to show that there exist $T \in \mathbb{R}_{\geq 0}$ such that $d(Gph(\sigma_1)|_T, Gph(\rho_1)|_T) < \epsilon$. Let us choose $\epsilon'$ as before. There exist $T \in \mathbb{R}_{\geq 0}$ such that $d(Gph(\sigma_2)|_T, Gph(\rho_2)|_T) < \epsilon'$. We show that the same $T$ works in $\mathcal{H}_1$ for the $\epsilon$. The proof is similar to showing that $d(\sigma_1, \rho_2) < \epsilon$ in the previous part (replace $Gph(\sigma_i)$ by $Gph(\sigma_i)|_T$). $\square$

The above theorem implies that the stability of a system $\mathcal{H}_1$ can be concluded by analysing a potentially simpler system $\mathcal{H}_2$ which uniformly continuously simulates $\mathcal{H}_1$.

*Remark 2.* Observe that the definition of stability crucially depends on the notion of distance between two executions. The definition used in this paper has been argued to be useful in [13] and accounts for all the discrete transitions in the execution. However, there might be situations where for stability purposes we might want to ignore the effects of these discrete transitions because these changes happen on a set of "measure 0" in the time domain. More precisely, let an execution $\sigma$ be a function with domain $[0, \infty)$ obtained by first dropping all the (discrete) transitions and then concatenating all the trajectories in order. The distance between two executions is then the supremum of the pointwise distance between their corresponding functions. It can be easily observed that the above proof of Theorem 1 works even for this definition of distance between executions. Thus, stability preservation by our definition of simulation is not very tightly bound to the specific definition of distance between executions.

As a corollary of Theorem 1, we obtain that Lyapunov stability and asymptotic stability are invariant under uniformly continuous bisimulations.

*Definition 11.* A *uniformly continuous bisimulation* between two *HTS*s $\mathcal{H}_1$ and $\mathcal{H}_2$ is a binary relation $R \subseteq S_1 \times S_2$ such that $R$ is a uniformly continuous simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $R^{-1}$ is a uniformly continuous simulation from $\mathcal{H}_2$ to $\mathcal{H}_1$.

Corollary 1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hybrid transition systems and $\mathcal{T}_1 \subseteq Exec(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq Exec(\mathcal{H}_2)$ be two sets of execution. Let $R \subseteq S_1 \times S_2$ be a uniformly continuous bisimulation between $\mathcal{H}_1$ and $\mathcal{H}_2$, and let $R$ be complete with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then the following hold:*

1. *$\mathcal{H}_1$ is Lyapunov stable with respect to $\mathcal{T}_1$ if and only if $\mathcal{H}_2$ is Lyapunov stable with respect to $\mathcal{T}_2$.*
2. *$\mathcal{H}_1$ is asymptotically stable with respect to $\mathcal{T}_1$ if and only if $\mathcal{H}_2$ is asymptotically stable with respect to $\mathcal{T}_2$.*

# 5. APPLICATIONS OF THE STABILITY PRESERVATION THEOREM

In this section, we show that various methods used in proving Lyapunov and asymptotic stability of systems can be formulated as constructing a simpler system which uniformly continuously simulates the original system and showing that the simpler system is Lyapunov or asymptotically stable, respectively.

## 5.1 Lyapunov Functions

We will show that Lyapunov's direct method for proving stability of dynamical systems can be interpreted as first constructing a simpler system using a Lyapunov function which uniformly continuously simulates the original system, and then establishing the stability of the simpler system. Then Theorem 1 gives us the stability of the original system.

Consider the following time-invariant system,

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and let $\bar{0}$ be an equilibrium point, that is, $f(\bar{0}) = \bar{0}$.

We can associate a hybrid transition system $\mathcal{H}_f = (S, \Sigma, \Delta)$ with the dynamical system in (3), where $S = \mathbb{R}^n$, $\Sigma = \emptyset$, $\Delta$ is the set of $C^1$ trajectories[3] $\tau : D \to \mathbb{R}^n$ (where $D \in \mathit{Int}$) such that $d\tau(t)/dt = f(\tau(t))$. Let the metric $d$ be the Euclidean distance. Let $\mathcal{T}_{f,x}$ be the set of all trajectories $\tau \in \Delta$ corresponding to an equilibrium point $x$, that is, $\tau$ such that $\tau(t) = x$ for all $t \in \mathit{Dom}(\tau)$.

Next we state Lyapunov's theorem which provides a sufficient condition for the stability of a system.

Theorem 2 (Lyapunov [20]). *Suppose that there exists a neighborhood $\Omega$ of $\bar{0}$ and a positive definite $C^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the algebraic condition:*

$$\dot{V}(x) \leq 0, \qquad \forall x \in \Omega, \tag{4}$$

*where $\dot{V}(x) = \frac{\partial V}{\partial x} f(x)$. Then System (3) is Lyapunov stable.*

*Furthermore, if $\dot{V}$ satisfies*

$$\dot{V}(x) < 0, \qquad \forall x \in \Omega/\{0\}, \tag{5}$$

*then System (3) is asymptotically stable.*

A $C^1$ positive definite function satisfying inequality (4) is called a *weak Lyapunov function* for $f$ over $\Omega$ and one satisfying (5) is called a *Lyapunov function*.

The following theorem formulates Lyapunov's first method as a stability preserving reduction to a simpler system using uniformly continuous simulations.

Theorem 3. *Let $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ be a dynamical system with an equilibrium point $\bar{0}$. Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ is a (weak) Lyapunov function for the dynamical system. Then:*

- *$V(\mathcal{H}_f)$ is (Lyapunov) asymptotically stable.*
- *$V$ restricted to a set containing a $\delta$-neighborhood of $\mathit{First}(\mathcal{T}_{f,\bar{0}})$ is a uniformly continuous simulation which is semi-complete with respect to $\mathcal{T}_{f,\bar{0}}$ and $V(\mathcal{T}_{f,\bar{0}})$.*

*Therefore, $\mathcal{H}_f$ is (Lyapunov) asymptotically stable.*

[3] $C^1$ is the set of continuously differentiable functions.

So Lyapunov's theorem can be casted as reducing the original system to a simpler system by uniformly upper continuous simulations and proving the stability of the simpler system. The steps in the above theorem give an alternate proof of Lyapunov and asymptotic stability using Theorem 1.

## 5.2 Multiple Lyapunov Functions

We show that proving stability of switched systems using multiple Lyapunov functions can be recast into the framework of Theorem 1.

A *switched system* consists of a set of dynamical systems and a switching signal which specifies the times at which the system switches its dynamics. Let us fix the following switched system with $N$ dynamical systems.

$$\dot{x} = f_i(x),\ i \in [N],\ x \in \mathbb{R}^n,$$
$$\alpha = (\{t_i\}_{i\in\mathbb{N}}, \{\omega_i\}_{i\in\mathbb{N}}), t_i \in \mathbb{R}_{\geq 0}, \omega_i \in [N]. \tag{6}$$

The *switching signal* $\alpha = (\{t_i\}_{i\in\mathbb{N}}, \{\omega_i\}_{i\in\mathbb{N}})$ is a monotonically increasing divergent sequence, that is, it satisfies $t_0 = 0$, $t_i < t_j$ for $j > i$ and for every $T \in \mathbb{R}_{\geq 0}$, there exists a $k$ such that $t_k > T$.

The solution of this system is the set of functions $\sigma : [0, \infty) \to \mathbb{R}^n$ such that $\sigma$ restricted to the interval between two switching times is a solution to the corresponding differential equation. Let $\sigma[a, b]$ denote the function from $[0, b-a]$ to $\mathbb{R}^n$ such that $\sigma[a, b](t) = \sigma(a + t)$. $\sigma$ is a solution of (6) if for every $i \in \mathbb{N}$, $\sigma[t_i, t_{i+1}]$ is a solution of the differential equation $\dot{x} = f_{\omega_i}(x)$.

We can associate a *HTS* $\mathcal{H}_{f_1,\cdots,f_N,\alpha}$ with the switched system in (6) given by $(S, \Sigma, \Delta)$, where $S = \mathbb{N} \times \mathbb{R}^n$, $\Sigma = \{((i, x), (i + 1, x)) \,|\, i \in \mathbb{N}, x \in \mathbb{R}^n\}$, and $\Delta$ consists of trajectories $\tau : [0, t_{i+1} - t_i] \to \{i\} \times \mathbb{R}^n$ for some $i \in \mathbb{N}$ such that there exists a trajectory $\theta : [0, t_{i+1} - t_i] \to \mathbb{R}^n$ which is a solution of the differential equation $\dot{x} = f_{\omega_i}(x)$, and the value of $\tau(t)$ is $(i, \theta(t))$, We can associate a metric $d$ over $S$, where $d((i, x), (j, y))$ is the Euclidean distance between $x$ and $y$ if $i = j$ and $\infty$ otherwise. Let $\mathcal{T}_{f_1,\cdots,f_N,\alpha,\bar{0}}$ be the set of all executions $\sigma$ in $\mathit{Exec}(\mathcal{H}_{f_1,\cdots,f_N,\alpha})$ such that $\mathit{States}(\sigma) \subseteq [N] \times \{\bar{0}\}$.

Next we state a result on the multiple Lyapunov method for stability analysis. Given a switching sequence $\alpha = (\{t_i\}_{i\in\mathbb{N}}, \{\omega_i\}_{i\in\mathbb{N}})$, we say that $t_i$ and $t_j$ are *adjacent* if $t_j$ is the first switching time after $t_i$ such that $\omega_i = \omega_j$.

Theorem 4 (Multiple Lyapunov Method [3]). *Let us consider the switched system of (6). Suppose there exist $N$ weak Lyapunov functions $V_i$ for $f_i$ over a neighborhood $\Omega$ of $\bar{0}$ such that for any pair of adjacent switching times $t_i$ and $t_j$, $V_{\omega_i}(\sigma(t_j)) \leq V_{\omega_i}(\sigma(t_i))$ for every solution $\sigma$ of the switched system. Then the switched system is Lyapunov stable.*

We call a vector of functions $\bar{V} = (V_1, \cdots, V_N)$ satisfying the hypothesis of Theorem 4, a *multiple Lyapunov function* for the switched system (6).

The above theorem can again be formulated as establishing a function from the *HTS* $\mathcal{H}_{f_1,\cdots,f_N,\alpha}$ to a simpler *HTS* using the functions $V_1, \cdots, V_N$ such that the simpler system is Lyapunov stable and the mapping is a uniformly continuous simulation, thereby proving the stability of the original system.

Given a vector of functions $\bar{V} = (V_1, \cdots, V_k)$, where $V_i : \mathbb{R}^m \to \mathbb{R}$ for $1 \leq i \leq k$, and a switching signal $\alpha = (\{t_i\}_{i\in\mathbb{N}}, \{\omega_i\}_{i\in\mathbb{N}})$, we define a function $\bar{V}[\alpha] : \mathbb{N} \times \mathbb{R}^m \to \mathbb{N} \times \mathbb{R}$, such that $\bar{V}[\alpha](i, x) = (i, V_{\omega_i}(x))$.

THEOREM 5. *Given the switched system in Equation (6), let $\bar{V}$ be a multiple Lyapunov function for the switched system. Then:*

- $\bar{V}[\alpha](\mathcal{H}_{f_1,\cdots,f_N,\alpha})$ *is Lyapunov stable.*
- $\bar{V}[\alpha]$ *when restricted to a set containing a $\delta$-neighborhood of First$(\mathcal{T}_{f_1,\cdots,f_N,\alpha,\bar{0}})$ is a uniformly continuous simulation which is semi-complete with respect to the sets $\mathcal{T}_{f_1,\cdots,f_N,\alpha,\bar{0}}$ and $\bar{V}[\alpha](\mathcal{T}_{f_1,\cdots,f_N,\alpha,\bar{0}})$.*

*Therefore, $\mathcal{H}_{f_1,\cdots,f_N,\alpha}$ is Lyapunov stable.*

## 5.3 Hartman-Grobman Theorem

We consider a theorem due to Hartman-Grobman which constructs linear approximations of non-linear dynamics and establishes a homeomorphism between the two dynamics. We show that the homeomorphic mapping from the non-linear dynamics to the linear dynamics is a uniformly continuous bisimulation. And hence one can use these reductions from non-linear to linear dynamics to potentially establish stability properties of non-linear dynamics by proving stability of the simpler linear dynamics, and using Theorem 1 to deduce the stability of the non-linear dynamics.

THEOREM 6 (LOCAL HARTMAN-GROBMAN THEOREM). *Consider a system $\dot{x} = F(x)$, where $F : \Omega \to \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ is an open set. Suppose that $x_0 \in \Omega$ is a hyperbolic equilibrium point of the system, that is, $A = DF(x_0)$ is a hyperbolic matrix, where $DF$ denotes the Jacobian of $F$. Let $\varphi$ be the (local) flow generated by the system, that is, $\varphi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a differentiable function such that $d\varphi(x,t)/dt = F(\varphi(x,t))$ for all $t \in \mathbb{R}_{\geq 0}$.*

*Then there are neighborhoods $U$ and $V$ of $x_0$ and a homeomorphism $h : U \to V$ such that $\varphi(h(x),t) = h(x_0 + e^{tA}(x - x_0))$ whenever $x \in U$ and $x_0 + e^{tA}(x - x_0) \in U$.*

Let us call a function $h$ satisfying the above condition, a *Hartman-Grobman* function associated with the dynamical system $\dot{x} = F(x)$. Given a *HTS* $\mathcal{H} = (S, \Sigma, \Delta)$, the restriction of $\mathcal{H}$ to a set $X \subseteq S$ is the *HTS* $(X, \Sigma \cap \mathit{Trans}(X), \Delta \cap \mathit{Traj}(S))$.

*Remark 3.* The terminologies referred to in the above theorem are standard. However, we define them in the Appendix for the sake of completeness.

THEOREM 7. *Let $\dot{x} = F(x)$ be a dynamical system, where $F : \Omega \to \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ is an open set, and let $x_0 \in \Omega$ be a hyperbolic equilibrium point. Let $G$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $G(x) = Ax$ where $A = DF(x_0)$. Let $h$ be a Hartman-Grobman function associated with the above dynamical system. Then, there exists a set $X$ containing a $\delta$-neighborhood of First$(T_{G,x_0})$ such that $h$ restricted to this set is a uniformly continuous bisimulation from $\mathcal{H}_G$ restricted to $X$ to $\mathcal{H}_F$ restricted to $h(X)$ and is complete with respect to $T_{G,x_0}$ and $T_{F,x_0}$.*

Again, we see that the reduction defined in Hartman-Grobman theorem from the non-linear dynamics to linear dynamics is a uniformly continuous bisimulation. We can use Theorem 7 along with Theorem 1 to deduce that $\mathcal{H}_F$ restricted to $h(X)$ is Lyapunov (asymptotically) stable iff $\mathcal{H}_G$ restricted to $X$ is Lyapunov (asymptotically) stable.

# 6. CONCLUSIONS

In this paper, we investigated pre-orders for reasoning about stability properties of dynamical and hybrid systems. We showed that bisimulation relations with continuity conditions, introduced in [6], are inadequate when stronger notions of stability like asymptotic stability, or the stability of trajectories is considered. We, therefore, introduced uniformly continuous simulations and bisimulations and showed that they form the semantic basis to reason about stability. Using such notions, we showed that, classical reasoning principles in control theory can be recast in a more "computer-science-like light", wherein they can be seen as being founded on abstracting/simplifying a system and then relying on the reflection of certain logical properties by the abstraction relation.

As argued in [6], one by-product of investigating the continuity requirements on simulations and bisimulations needed to reason about stability, is that it allows one to conclude the inadequacy of the modal logic in [8] to express stability properties. What is the right logic to express properties like stability? That remains open. Just like Hennessy-Milner logic serves as the logical foundation for classical simulation and bisimulation, the right modal logic that can express stability might form the logical basis for the simulation and bisimulation relations introduced here.


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# APPENDIX

## A. COMPARISON WITH PREVIOUS DEFINITIONS

We begin by showing that the notion of simulation and bisimulation introduced in [6] are not sufficient to reason about stability of trajectories even with an additional constraint of continuity on the relation.

### *Lyapunov Stability of Trajectories.*

Consider a hybrid transition system $\mathcal{H}_1 = (S_1, \Sigma_1, \Delta_1)$, where

- the state space $S_1$ is the set $\mathbb{R}^2_{\geq 0}$, which is the positive quadrant of the two dimensional plane;
- the set of transitions $\Sigma_1$ is the empty set; and
- $\Delta_1$ is the set $\{f_m \,|\, m \in \mathbb{R}_{\geq 0}\}$, where for a particular $m \in \mathbb{R}_{\geq 0}$, $f_m : [0, \infty) \to \mathbb{R}^2_{\geq 0}$ is the trajectory such that $f(t) = (t, m)$.


**Figure 1: A *HTS* which is Lyapunov stable**

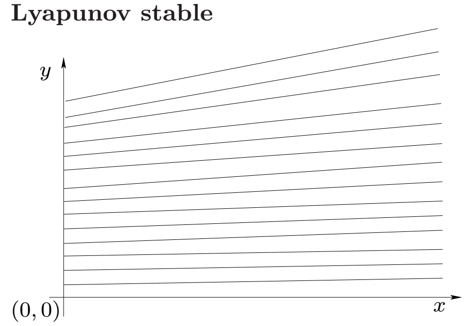


**Figure 2: An unstable** ***HTS***

As shown in Figure 1, $\mathcal{H}_1$ consists of trajectories which start on the positive $y$-axis and evolve parallel to the positive $x$-axis. It is easy to see that $\mathcal{H}_1$ is Lyapunov stable with respect to the unique trajectory $\tau_1$ which starts at the origin and moves along the $x$-axis.

Now let us consider another system $\mathcal{H}_2 = (S_2, \Sigma_2, \Delta_2)$, shown in Figure 2, which is similar to $\mathcal{H}_1$, that is, $S_2 = S_1$ and $\Sigma_2 = \Sigma_1$, except that $\Delta_2 = \{f_m : [0, \infty) \to \mathbb{R}_{\geq 0} \,|\, f_m(t) = (t, m(1+t)), m \in \mathbb{R}_{\geq 0}\}$. The trajectories of $\mathcal{H}_2$ start on the positive $y$-axis and evolve along a straight line whose slope is given by the $y$ intercept. So they form a diverging set of straight lines. Consider the trajectory $\tau_2$ which starts at the origin and evolves along the $x$-axis. Note that $\mathcal{H}_2$ is not Lyapunov stable with respect to $\{\tau_2\}$.

However, $R = \{((x_1, y_1), (x_2, y_2)) \,|\, x_1 = x_2 \text{ and } y_2 = y_1(1 + x_1)\}$ is a bisimulation relation between $\mathcal{H}_1$ and $\mathcal{H}_2$, in fact, a bi-continuous bijection, that is, $R$ and $R^{-1}$ (when considered as single valued functions) are continuous. Thus, Lyapunov stability with respect to trajectories is not invariant under the bisimulations that are only continuous.

### *Asymptotic Stability of Trajectories.*

We can show in a similar fashion that bi-continuous bisimulations do not preserve asymptotic stability. For example, consider a system $\mathcal{H}_3$ which is similar to $\mathcal{H}_2$ except that $f_m$

is defined as $f_m(t) = (t, me^{-t})$. Note that $\mathcal{H}_1$ is not asymptotically stable, where as $\mathcal{H}_3$ is. And there is a bi-continuous bisimulation relation given by $R = \{((x_1, y_1), (x_2, y_2)) \,|\, x_1 = x_2 \text{ and } y_2 = y_1 e^{-x_1}\}$.

*Insufficiency of the continuity conditions in [6] for asymptotic stability with respect to a set of points.*

Consider a system $\mathcal{T}_1$ with statespace $\mathbb{R}$ and the equilibrium point 0. Let the trajectory $x$ starting at any point $x(0) > 0$ in $\mathbb{R}$ be such that $x(t) > 0$ for all $t$, $x(t_1) > x(t_2)$ for all $t_1 < t_2$ and $x(t) \to 0$ as $t \to \infty$. Similarly, a trajectory $x$ starting at any point $x(0) < 0$ in $\mathbb{R}$ be such that $x(t) < 0$ for all $t$, $x(t_1) < x(t_2)$ for all $t_1 < t_2$ and $x(t) \to 0$ as $t \to \infty$. Note that $\mathcal{T}_1$ is asymptotically stable with respect to the equilibrium 0.

Next consider a system $\mathcal{T}_2$ with statespace $\mathbb{R}$ such that every points is an equilibrium point. Note that $\mathcal{T}_2$ is stable with respect to 0, but not asymptotically stable.

Now, we define a relation $R$ from $\mathcal{T}_1$ to $\mathcal{T}_2$ satisfying the hypothesis of Theorem 2 of [6]. $R = \{(x, y) \in \mathbb{R}^2 \,|\, y \geq x > 0$ or $0 < x \leq y\}$. Then $R$ satisfies the following:

- $R^{-1}$ is a simulation, that is, if $x_1$ can go to $x_2$ in time $t$ in $\mathcal{T}_2$, then for every $y_1$ such that $(x_1, y_1) \in R^{-1}$, there exists a $y_2$ such that $y_2$ can be reached from $y_1$ in time $t$ and $(x_2, y_2) \in R^{-1}$,
- $R$ is upper semi-continuous, and
- $R^{-1}$ is lower semi-continuous, that is for any open set $X$, $R(X) = \{y \,|\, \exists x \in X : (x, y) \in R\}$ is an open set.

It can be verified that $R$ satisfies the above conditions and hence the hypothesis of Theorem 2 in [6]. However, the conclusion of the theorem does not hold for asymptotic stability because it would state that if $\mathcal{T}_1$ is asymptotically stable with respect to a closed set $S$, then $\mathcal{T}_2$ is asymptotically stable with respect to $R(S)$. Note however that $\mathcal{T}_1$ is asymptotically stable with respect to $\{0\}$, however $\mathcal{T}_2$ is not asymptotically stable with respect to $R(\{0\}) = \{0\}$.

# B. PRELIMINARIES AND PROOFS OF THEOREMS

In this section, we recall certain standard definitions and provide proofs of theorems in Section 5.

## B.1 Preliminaries

Consider a $C^1$ (i.e., continuously differentiable) function $V : \mathbb{R}^n \to \mathbb{R}$. It is called *positive definite* if $V(\bar{0}) = 0$ and $V(x) > 0$ for all $x \neq 0$. Let

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x),$$

and note that $\dot{V}$ is the time derivative of $V(x(t))$, where $x(t)$ is a solution of the Equation 3.

Below, we present the definitions of terminologies used in Theorem 6. A function $f : A \to B$, where $A, B \subseteq \mathbb{R}^n$ is a *homeomorphism* if $f$ is a bijection and both $f$ and $f^{-1}$ are continuous. A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is given by $m$-real valued component functions, $y_1(x), \cdots, y_m(x)$, where $x = (x_1, \cdots, x_n)$. The partial derivatives of all these functions (if they exist) can be organized in a $m \times n$ matrix called the *Jacobian* of $F$, denoted by $DF(x)$, where the entry in the $i$-th row and $j$-th column is $\partial y_i / \partial x_j$.

Given an $n$-vector $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$, $DF(a)$ is the matrix obtained by substituting $x_i$ in the terms of the matrix $DF(x_1, \cdots, x_n)$ by $a_i$. A square matrix $A$ is *hyperbolic* if none of its eigen values are purely imaginary values (including 0).

## B.2 Proof of Theorem 3

PROOF. (Sketch.) Proof of Part (1): Let $\mathcal{H}_f = (S_1, \Sigma_1, \Delta_1)$, and $V(\mathcal{H}_f) = (S_2, \Sigma_2, \Delta_2)$. We need to show that if $V$ is a weak Lyapunov function, then $V(\mathcal{H}_f)$ is Lyapunov stable, and if $V$ is a Lyapunov function, then $V(\mathcal{H}_f)$ is asymptotically stable.

Let $\Omega$ be an open subset around $\bar{0}$ as given by Lyapunov's theorem. Observe that if $V$ is a weak Lyapunov function, then for every $\tau \in \Delta_2$ such that $\tau(0) \in V(\Omega)$, for any $t_1 < t_2$, $\tau(t_1) \geq \tau(t_2)$, since $\dot{V}(x) \leq 0$ for every $x \in \Omega$ and $\tau$ arises from a trajectory of $\mathcal{H}_f$. Hence the distance of $\tau(t)$ from 0 is non-increasing as time $t$ progresses. Since for any $\tau^* \in V(\mathcal{T}_{f,\bar{0}})$, $\tau^*(t) = 0$ for any $t$, $d(\tau(t), \tau^*(t)) \leq d(\tau(0), \tau^*(0))$. So given any $\epsilon > 0$, choose a $\delta'$ which is less than $\delta$ and $\epsilon$. Then $V(\mathcal{H}_f)$ is Lyapunov stable with respect to $\delta'$, that is, any $\tau$ starting with in $\delta'$-neighborhood of *First*$(V(\mathcal{T}_{f,\bar{0}}))$ (which is same as $V($*First*$(\mathcal{T}_{f,\bar{0}}))$ in this case) remains with in a distance of $\epsilon$ from some trajectory in $V(\mathcal{T}_{f,\bar{0}})$.

Next, if $V$ is a Lyapunov function, we need to show that in addition to the above, there is a neighborhood of *First*$(V(\mathcal{T}_{f,\bar{0}}))$ such that trajectories starting from it converge to some trajectory in $V(\mathcal{T}_{f,\bar{0}})$. Let $\Omega$ be the set associated with the Lyapunov function $f$ such that $\dot{V}(x) < 0$ for all $x \in \Omega$. We will show that $V(\Omega)$ is such a neighborhood. It suffices to show that for any trajectory $\tau : [0, \infty) \to S_2$ starting from a state $x \in V(\Omega)$, $\tau$ converges to 0. Since $V$ is positive and decreasing along any solution $\tau$, it has a limit $c \geq 0$ as $t \to \infty$. If $c = 0$, then we are done. Otherwise, the solution cannot enter the set $\{x : V(x) < c\}$. In this case, the solution evolves in a compact set that does not contain the origin. Let the compact set be $C$. Let $d = max_{x \in C} \dot{V}(x)$; this number is well defined due to compactness of $C$and negative due to 5. We have $\dot{V} \leq d$, and hence $V(t) \leq V(0) + dt$. But then $V$ will eventually become smaller than $c$, which is a contradiction.

Proof of Part (2): $V$ restricted to a set containing a $\delta$-neighborhood of *First*$(\mathcal{T}_{f,\bar{0}})$ is a uniformly continuous simulation. First observe that $V$ is a simulation relation from $\mathcal{H}_f$ to $V(\mathcal{H}_f)$ by definition, and $V$ is an upper semi-continuous function (since it is continuous). Therefore its inverse $V^{-1}$ is also upper semi-continuous. Further, $\Omega$ is an open set around $\bar{0}$, hence $V(\Omega)$ is an open set around $V(\bar{0}) = 0$. Let $\delta$ be such that a closed ball of radius $\delta$ around 0 is contained in $V(\Omega)$. Then $V$ restricted to the compact set, closed ball of radius $\delta$ around 0, is a uniformly continuous function and hence so is $V^{-1}$. Note that $V$ restricted to the compact set is also a simulation owing to $V$ being a decreasing function. $V$ is semi-complete, since the first three conditions are trivially true and the fourth condition is true because of the previous observation.

Conclusion: Let $V$ restricted to the closed ball of radius $\delta$ be the function $V'$. Since $V'$ is a uniformly continuous simulation between $\mathcal{H}_1 = \mathcal{H}_f$ and $\mathcal{H}_2 = V(\mathcal{H}_f)$ and is semi-complete with respect to $\mathcal{T}_1 = \mathcal{T}_{f,\bar{0}}$ and $\mathcal{T}_2 = V(\mathcal{T}_2)$; and $\mathcal{H}_2$ is (Lyapunov) asymptotically stable, it follows from Theorem 1 that $\mathcal{H}_f = \mathcal{H}_1$ is (Lyapunov) asymptotically stable. $\square$