On the Decidability of Stability of Hybrid Systems

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ABSTRACT

A rectangular switched hybrid system with polyhedral invariants and guards, is a hybrid automaton in which every continuous variable is constrained to have rectangular flows in each control mode, all invariants and guards are described by convex polyhedral sets, and the continuous variables are *not* reset during mode changes. We investigate the problem of checking if a given rectangular switched hybrid system is stable around the equilibrium point 0. We consider both Lyapunov stability and asymptotic stability. We show that checking (both Lyapunov and asymptotic) stability of planar rectangular switched hybrid systems is decidable, where by planar we mean hybrid systems with at most 2 continuous variables. We show that the stability problem is undecidable for systems in 5 dimensions, i.e., with 5 continuous variables.

Categories and Subject Descriptors

D.2 [**Software**]: Software Engineering

General Terms

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1. INTRODUCTION

Stability is a key property required of many dynamical and hybrid systems. Intuitively, it says that when a system is started somewhere close to its desired operating behavior, it will stay close to its desired operating behavior at all times (Lyapunov stability) and will converge to its desired operating behavior in the limit (asymptotic stability). Of particular importance is the stability of equilibrium points, which requires that a system starting close to an equilibrium point stays close to the equilibrium point at all times (Lyapunov stability) and converges to the equilibrium point in the limit (asymptotic stability).

Given the importance of system designs to be stable, a number of proof techniques have been discovered to establish the stability of a system under a controller. These rely on extensions of Lyapunov's second or direct method of establishing stability, wherein one identifies an "energy" function that decreases over the trajectories of the system. In the context of hybrid systems, this could be either a common function for all control modes, or separate functions for each control mode that nonetheless decrease over finite sequences of control mode changes. For a comprehensive overview of these techniques, see surveys [5, 7, 6] and textbooks [15, 11]. Such Lyapunov functions are either constructed through the ingenuity of the designer or by algorithmically searching for Lyapunov functions of a special form using convex programming. Automated techniques have also been developed to check a weaker, "qualitative" version of stability, wherein one only requires that all trajectories starting from an initial state eventually enter a "good" region [12, 13, 8, 9]. All the above techniques typically only check for sufficient conditions for a system to be stable. In other words, if the above approaches (automatic and manual) succeed then one can conclude that the system is stable. On the other hand, if they fail then one *cannot* conclude the instability of the system.

Unfortunately, despite the undeniable importance of stability, very little is known about the *computational difficulty* of deciding stability of a hybrid system. This is quite unlike other important properties like safety and liveness. In a series of papers [3, 4, 2], Blondel, Tsitsiklis, and their co-authors, present some results concerning discrete-time, switched hybrid systems with linear dynamics in each control mode, i.e., in each control mode the state at time instant $k+1$ (x_{k+1}) is given by $x_{k+1} = Ax_k$, where A depends on the current mode, and mode switches don't change the continuous state x. They show that stability is undecidable for saturated linear dynamical systems [2], is NP-hard for various restricted forms of discrete time linear switched hybrid systems with mode switches governed by polyhedral guards [4] (the upper bound in these cases is not known, and the problems may in fact be undecidable), and undecidable for certain simple systems with a control input [4].

In this paper, we present some results concerning decidability of stability of equilibrium points in the continuous-time, hybrid setting; we hope this will spur much needed fur-

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ther research into the computational complexity of stability. We consider *rectangular switched hybrid systems with polyhedral guards and invariants*. These are hybrid automata that are switched (i.e., the continuous state does not change when the control mode changes), the continuous dynamics in each control mode is governed by rectangular constraints which require that the time derivative of any variable's continuous trajectory lie in an interval at all times, and the invariants for each control mode and guards on all control switches are given by convex polyhedra. Thus, these systems are generalizations of piecewise constant derivative (PCD) systems [1], and related to rectangular hybrid automata in the sense that guards and invariants can be more general than rectangular sets, but restricted in that control mode switches don't reset the continuous state.

Our main result is that the stability (both Lyapunov and asymptotic) of planar (i.e., systems with only two continuous variables) rectangular switched hybrid systems with polyhedral invariants and guards can be effectively decided. Our proof relies on the following sequence of observations. We assume that 0 is the desired equilibrium point whose stability we wish to establish. We first observe that a hybrid system $\mathcal{H}$ is stable at 0 iff for some $\gamma > 0$, $\mathcal{H}_\gamma$ is stable, where $\mathcal{H}_\gamma$ is a system that is identical to $\mathcal{H}$ in the open ball of radius γ around 0. For the class systems we consider, $\mathcal{H}_\gamma$ has a special structure wherein all the guards and invariants can be seen to be unions of "wedge"-like sets. Based on this observation, we show that stability questions about $\mathcal{H}_\gamma$ can be reduced to questions about an "equivalent" finite, weighted graph $G_\mathcal{H}$. The crux of the decidability proof is showing that this graph $G_\mathcal{H}$ can be effectively constructed from $\mathcal{H}$. This involves showing that some restricted forms of reachability predicates can be answered for the class of planar switched systems we consider. These observations about the computability of limited reachability predicates for our systems may be of independent interest, since the decidability of the general reachability problem for our class of hybrid systems is not known. We conclude our results by showing that stability is indeed likely to be a difficult problem: we show that determining the Lyapunov stability (at 0) of a rectangular switched hybrid system (with polyhedral invariants and guards) with at least 5 continuous variables, is undecidable.

Our undecidability result for 5 dimensions may seem unsurprising and disappointing in the light of the fact that reachability is undecidable for 3-dimensional PCD systems (which is a special case of our systems). This disappointment stems from a popular belief that stability is "harder" than reachability. We provide some evidence to support this belief. Our undecidability proof reduces reachability in certain special cases to Lyapunov stability. On the other hand, our paper also provides some evidence to suggest that establishing the hardness of stability relative to reachability (in general) may not be as obvious as the common belief seems to suggest. First, the reachability problem is not known to be decidable for the class of planar systems we consider, even though we show stability to be decidable; for special subclasses of the hybrid systems considered here like PCD [1] and monotonic linear systems [14], reachability is known to be decidable. Next, while our stability algorithm does use certain types of reachability queries, it does so for special "pie shaped" hybrid systems. The reason is because we show that for stability one can always focus on the behavior of the system close to 0, and in all systems sufficiently close to 0, the invariants and guards look like unions of "wedge-like" sets. Thus, the reachability problem must be hard on such instances for the hardness to be lifted to stability.

2. PRELIMINARIES

2.1 Notations

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the set of reals, non-negative reals, rationals and natural numbers, respectively. Given a function F, we use $Dom(F)$ to denote the domain of F. Given function $F : A \to B$ and a set $A' \subseteq A$, $F(A')$ will denote the set $\{F(a) \,|\, a \in A'\}$.

Sequences.

Let *SeqDom* denote the set of all subsets of $\mathbb{N}$ which are prefix closed, where a set $S \subseteq \mathbb{N}$ is prefix closed if for every $m, n \in \mathbb{N}$ such that $n \in S$ and $m < n$, $m \in S$. For a finite $S \in$ *SeqDom*, we use $|S|$ to denote the largest element of S, that is, for $S = \{0, 1, \cdots, n\}$, $|S| = n$. A *sequence* over a set A is a mapping from an element of *SeqDom* to A. Given a sequence $\pi : S \to A$, we also denote the sequence by enumerating its elements in the order, that is, $\pi(0), \pi(1), \cdots$.

Convex polyhedral sets.

Let X denote the d-dimensional Euclidean space $\mathbb{R}^d$ for some d. Given $\mathbf{x}, \mathbf{y} \in X$, we use the standard notation $\mathbf{x} \cdot \mathbf{y}$ to denote the dot product of vectors $\mathbf{x}$ and $\mathbf{y}$ and $|\mathbf{x}|$ to denote the Euclidean norm. For a vector $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $1 \leq i \leq d$, $(\mathbf{x})_i$ will denote the projection of $\mathbf{x}$ to the i-th component, that is, x_i. Given $\epsilon \in \mathbb{R}_{\geq 0}$, we use $B_\epsilon(\mathbf{x})$ to denote an open ball around $\mathbf{x}$ of radius ϵ, that is, $B_\epsilon(\mathbf{x}) = \{\mathbf{y} \,|\, |\mathbf{x} - \mathbf{y}| < \epsilon\}$. A set $S \subseteq X$ is *open* if for every $\mathbf{x} \in S$, there exists a $\delta > 0$ such that $B_\delta(\mathbf{x}) \subseteq S$.

A set $S \subseteq X$ is *convex* if for every $\mathbf{x}, \mathbf{y} \in S$ and $\alpha \in [0, 1]$, $\alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \in S$. Let *Conv*$(X)$ denote the set of all convex subsets of X. Given a set S, *ConvHull*(S) is the smallest convex set containing S. A *half-space* in X is the set which can be expressed as the set of all points $\mathbf{x} \in X$ satisfying a linear constraint, $\mathbf{a} \cdot \mathbf{x} + \mathbf{b} \sim 0$, for some $\mathbf{a}, \mathbf{b} \in X$ and $\sim \in \{<, \leq\}$. A *convex polyhedral set* is an intersection of finitely many half-spaces. We will use *ConvPolyhed*(X) to denote the set of all convex polyhedral subsets of X. Given a convex polyhedral set S and a point $s \in S$, we call the cone of S at s, the set of all vectors $v \neq \mathbf{0}$ such that there exists a $t > 0$, for which $s + vt \in S$. We denote this set by *Cone*(S, s).

A *partition* $\mathcal{P}$ of $\mathbb{R}^d$ into convex polyhedral sets is a finite set of convex polyhedral sets $\{P_1, \cdots, P_k\}$ such that $\cup_{i=1}^k P_i = \mathbb{R}^d$ and for each $i \neq j$, $P_i \cap P_j = \emptyset$.

Intervals and Rectangular sets.

An interval is a convex subset of $\mathbb{R}$ and is represented using the standard notation ($[a, b]$, (a, b), and so on). An interval I is said to be compact if $I = [a, b]$ for some $a, b \in \mathbb{R}$. We use *TimeDom* to represent the finite and infinite closed intervals starting from 0, that is, *TimeDom* consists of the intervals $I = [0, T]$ for some $T \in \mathbb{R}_{\geq 0}$ and the interval $[0, \infty)$. Given $I \in$ *TimeDom*, we define the size of the interval I, denoted *Size*(I), to be an element in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that for a compact interval $I = [0, T]$, *Size*$(I) = T$, otherwise *Size*$(I) = \infty$.

A *rectangular set* over X is a set $R \subseteq X$ which can be expressed as the Cartesian product of compact intervals, that is, R is given by a set $I_1 \times \cdots \times I_d$, where I_i for $1 \leq i \leq d$ are compact intervals. Let $Rect(X)$ denote the set of all rectangular subsets of X. Given $R \in Rect(\mathbb{R}^d)$, we define the extreme vectors in R, denoted $Extremes(R)$, as $Extremes(R) = \{\mathbf{v} \in R \,|\, \forall i, (\mathbf{v})_i = min((R)_i) \text{ or } max((R)_i)\}$.

Planar Elements.

Two dimensional objects will be referred to using the prefix "planar". A planar convex polyhedral set can be decomposed into an interior region and boundary elements which are the vertices and the straight line segments. Formally, given a planar convex polyhedral set S, *region* of S is the set of all points in the interior of S, that is, all point $s \in S$ such that $B_\delta(s) \subseteq S$ for some $\delta > 0$. *Boundary* of S is the set of all points $s \in S$ such that s is not in the interior of S. A *line* of S is a maximal convex subset l contained in the boundary of S, such that for every $s \in I$, there exist $s_1, s_2 \in I$ and $0 < \lambda < 1$ with $s = \lambda s_1 + (1-\lambda)s_2$. (The last condition is to eliminate the end-points of the line, if any). A *vertex* of S is a singleton set $\{s\}$ such that s belongs to the boundary of S but is not contained in any line of S. The closure of S, denoted $Closure(S)$, is the smallest closed set containing S. Elements of S, denoted $Elements(S)$, is the set of all vertices, lines and regions of S.

Let $\mathcal{P} = \{P_1, \cdots, P_k\}$ be a convex polyhedral partition of $\mathbb{R}^2$. Elements of $\mathcal{P}$, denoted by $Elements(\mathcal{P})$, is the set of elements of the convex polyhedral sets in $\mathcal{P}$, that is, $Elements(\mathcal{P}) = \cup_{i=1}^{k} Elements(P_i)$.

Graphs.

A *graph* G is a triple (V, E, V_0), where V is a finite set of nodes, $E \subseteq V \times V$ is a finite set of edges, and $V_0 \subseteq V$ is a finite set of initial nodes. A *path* π of a graph $G = (V, E, V_0)$ is a finite sequence of nodes $v_0, \cdots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $0 \leq i < k$. The length of a path π, denoted $|\pi|$, is the number of edges occurring in it. A path π is *simple* if all the nodes occurring in the path are distinct. A *cycle* in a graph $G = (V, E, V_0)$ is a path $\pi = v_0, \cdots, v_k$ in which the first and the last nodes are the same, that is, $v_0 = v_k$. A cycle is *simple* if all the nodes except the last one are distinct. A node v is *reachable* from a node u if there exists a path whose first element is u and the last element is v, that is, there exists a π such that $\pi(0) = u$ and $\pi(|Dom(\pi)|) = v$.

We associate weights with the edges of a graph using weighting functions. A *weight function* w of a graph $G = (V, E, V_0)$ is a function $w : E \to \mathbb{R} \cup \{-\infty, +\infty\}$. We extend the weight function to a path as follows. Weight of a path π of G, denoted $w(\pi)$, is $\sum_{0 \leq i < |Dom(\pi)|} w((\pi(i), \pi(i+1)))$.

A *strongly connected component* SCC in a graph $G = (V, E)$ is a set of nodes $V' \subseteq V$ such that for every $v_1, v_2 \in V'$, there exists a path from v_1 to v_2 in G such that all elements in the path are in V'. Let $SCC(G)$ denote the set of maximal strongly connected components of G. Note that $SCC(G)$ represents a partition of the nodes in V, and there is a partial order on the elements of $SCC(G)$, where an element C_1 is less than C_2 if there exists a path from some node in C_1 to some node in C_2. In particular, we can construct a graph $G/_{SCC}$ whose nodes are the elements of $SCC(G)$ and whose edges are pairs (C_1, C_2) such that there exists $u \in C_1$ and $v \in C_2$ with (u, v) being an edge of G. We call this the quotient graph of G with respect to strongly connected components.

2.2 Switched Hybrid Systems

In this section, we define a formal model of systems which exhibit discrete-continuous behaviors. We focus on a class of systems, which exhibit different continuous behaviors in different modes of the system, and in which the switching between the modes is dictated by a controller which can read the continuous state of the system. In particular, the controller cannot force resets in the system state. We use the hybrid automaton model of [10], with some restrictions on the class of constraints used to define the elements of the model.

A *switched hybrid system* (*SHS*) of dimension d is given by $\mathcal{H} = (Q, Q_0, \Delta, X, Fl, Inv, Gd)$, where

- Q is a finite set of locations or modes,
- $Q_0 \subseteq Q$ is a set of initial locations,
- $\Delta \subseteq Q \times Q$ is a set of mode changes or transitions,
- $X = \mathbb{R}^d$ is the set of continuous states,
- $Fl : Q \to Conv(X)$ specifies vector fields for locations,
- $Inv : Q \to Conv(X)$ specifies invariants for locations, and
- $Gd : \Delta \to Conv(X)$ specifies guards for transitions.

In this paper, we focus on only switched hybrid systems, hence, we take the liberty to drop the prefix "switched" when referring to this class of systems.

Notation. The components of a hybrid system $\mathcal{H}_r$ are denoted using appropriate subscripts, for example, the set of locations of $\mathcal{H}_r$ is denoted by Q_r.

The hybrid system $\mathcal{H}$ starts in a state in X with the control in a mode in Q_0. The state of the system then evolves such that the derivative of the evolution belongs to the flow Fl of the current mode, and simultaneously satisfies the invariant of the mode. The system can change mode at any time from the current mode q_1 to a new mode q_2 provided there is a transition (q_1, q_2) in Δ and the current state satisfies the guard associated with the transition.

The semantics of a switched hybrid system $\mathcal{H}$ is given by the set of executions of the system. An *execution* of $\mathcal{H}$ is a triple $\eta = (\sigma, \tau, \gamma)$ such that there exists a $D \in TimeDom$ and an $S \in SeqDom$, which satisfy the following:

- $\sigma : D \to X$ is a continuous function,
- $\tau : S \to D$ is a non-decreasing function, such that
  - $\tau(0) = 0$,
  - if $S = \mathbb{N}$, then $\lim_{i \to \infty} \tau(i) = \infty$,
  - σ is differentiable in the interval $(\tau(i), \tau(i+1))$ for every i such that $\tau(i) < \tau(i+1)$, and in the interval $(\tau(|S|), Size(D))$ if S is finite, and
- $\gamma : S \to Q$, such that
  - for every i such that $\tau(i) = \tau(i+1)$, $e = (\gamma(i), \gamma(i+1)) \in \Delta$ and $\sigma(\tau(i)) \in Gd(e)$,
  - for every i such that $\tau(i) < \tau(i+1)$, $\gamma(i) = \gamma(i+1)$, $\frac{d\sigma}{dt}(t) \in Fl(\gamma(i))$ and $\sigma(t) \in Inv(\gamma(i))$, for every $t \in (\tau(i), \tau(i+1))$, and

– if S is finite and $\tau(|S|) < \mathit{Size}(D)$, then again $\frac{d\sigma}{dt}(t) \in \mathit{Fl}(\gamma(|S|))$ and $\sigma(t) \in \mathit{Inv}(\gamma(|S|))$, for every $t \in (\tau(|S|), \mathit{Size}(D))$.

Further, if $\gamma(0) \in Q_0$, then we call $\eta = (\sigma, \tau, \gamma)$ an *initial execution* of $\mathcal{H}$.

Notation. Given an execution η_r, we refer to its components with appropriate subscripts, that is, $\eta_r = (\sigma_r, \tau_r, \gamma_r)$.

An execution η of $\mathcal{H}$ is called *complete* if $\mathit{Size}(\mathit{Dom}(\sigma)) = \infty$, and *incomplete* or *finite* otherwise. We denote the set of executions of $\mathcal{H}$ by $\mathit{Exec}(\mathcal{H})$, the set of initial executions of $\mathcal{H}$ by $\mathit{IExec}(\mathcal{H})$ and the set of complete executions of $\mathcal{H}$ by $\mathit{CExec}(\mathcal{H})$.

η is said to be an execution from $\mathbf{x}$ to $\mathbf{y}$ if σ is an incomplete execution with $\sigma(0) = \mathbf{x}$ and $\sigma(\mathit{Size}(\mathit{Dom}(\sigma))) = \mathbf{y}$. Further, it is an execution from $(q_1, \mathbf{x})$ to $(q_2, \mathbf{y})$, if in addition $\gamma(0) = q_1$ and $\gamma(|\mathit{Dom}(\gamma)|) = q_2$. Also, η *reaches* $\mathbf{y}$ ($(q, \mathbf{y})$) if η is an execution from some $\mathbf{x}$ to $\mathbf{y}$ (and $\gamma(|\mathit{Dom}(\gamma)|) = q$), and η *touches* $\mathbf{y}$ ($(q, \mathbf{y})$) if there exists a time $t \in \mathit{Dom}(\sigma)$ (and $i \in \mathit{Dom}(\tau)$) such that $\sigma(t) = \mathbf{y}$ (and $\tau(i) = t$ and $\gamma(i) = q$).

Splitting an execution.

Next we define splitting an execution into two executions such that executing the second from the end-point of the first results in the original execution. Two executions η_1, η_2 *split* an execution η if the following hold:

- There exists $c \in \mathbb{R}_{\geq 0}$ such that $c = \mathit{Size}(\mathit{Dom}(\sigma_1))$, and $\mathit{Dom}(\sigma_1) \cup \{c + t \,|\, t \in \mathit{Dom}(\sigma_2)\} = \mathit{Dom}(\sigma)$.
- $|\mathit{Dom}(\tau)| = |\mathit{Dom}(\tau_1)| + |\mathit{Dom}(\tau_2)| - 1$, such that, for every $i \in \mathit{Dom}(\tau_1)$, $\tau_1(i) = \tau(i)$, for every $j > |\mathit{Dom}(\tau_1)|$ such that $j \in \mathit{Dom}(\tau)$, $\tau_2(j - |\mathit{Dom}(\tau_1)| + 1) = \tau(j) - c$, and $\tau_2(0) = c$.
- $\gamma_1(i) = \gamma(i)$ for all $i \in \mathit{Dom}(\gamma_1)$, for every $j > |\mathit{Dom}(\gamma_1)|$ such that $j \in \mathit{Dom}(\gamma)$, $\gamma_2(j - |\mathit{Dom}(\gamma_1)| + 1) = \gamma(j)$, and $\gamma(0) = \gamma_1(|\mathit{Dom}(\gamma_1)|)$.

A finite sequence $\eta_1, \cdots, \eta_n$ splits η, if there exists a sequence of sequences starting from the sequence η and ending with the sequence $\eta_1, \cdots, \eta_n$ such that every successive sequence is obtained from the previous one by splitting some execution. An infinite sequence $\eta_1, \eta_2, \cdots$ splits η, if for every $i \geq 1$, there exists an η'_i such that the finite sequence $\eta_1, \eta_2, \cdots, \eta_i, \eta'_i$ splits η.

Note that in general to split an η uniquely we need to specify the time of splitting σ and a element in the $\mathit{Dom}(\tau)$ which maps to the time of splitting, if one exists. $\eta[t_1, t_2]$ will used to denote η_2 in some splitting η_1, η_2, η_3 of η such that $t_1 = \mathit{Size}(\mathit{Dom}(\sigma_1))$ and $t_2 - t_1 = \mathit{Size}(\mathit{Dom}(\sigma_2))$. A prefix of η is an execution $\eta[0, t]$ for some $t \in \mathit{Dom}(\sigma)$.

Remark. Note that if η is an execution of $\mathcal{H}$, then $\eta[t_1, t_2]$ exists and is also an execution of $\mathcal{H}$.

Scaling and Weight.

Given a finite execution η, we define the *scaling* and *weight* of an execution η, denoted $\mathit{Scaling}(\eta)$ and $\mathit{Weight}(\eta)$, to be $|\sigma(\mathit{Size}(\mathit{Dom}(\sigma)))|/|\sigma(0)|$ and $\ln(\mathit{Scaling}(\eta))$, respectively.

PROPOSITION 1. *Given a splitting $\eta_1, \cdots, \eta_n$ of a finite execution η, weight of η is the sum of the weights of η_i for $1 \leq i \leq n$.*

Restrictions of Hybrid Systems.

Let $\mathcal{H}$ be a hybrid system. Given a set of locations $S \subseteq Q$, the restriction of the $\mathcal{H}$ to S, is the system $(Q \cap S, Q_0 \cap S, \Delta \cap S \times S, X, \mathit{Fl}[S], \mathit{Inv}[S], \mathit{Gd}[S])$, where $f[A]$ is the function whose domain is $A \subseteq \mathit{Dom}(f)$ and agrees with f on all points in the domain. We will denote this system as $\mathcal{H}[Q \to S]$. Similarly, given a set $S \subseteq X$, we define the restriction of $\mathcal{H}$ to the set S, denoted $\mathcal{H}[X \to S]$ as the system $(Q, Q_0, \Delta, X, \mathit{Fl}', \mathit{Inv}', \mathit{Gd}')$, where $\mathit{Inv}'(q) = \mathit{Inv}(q) \cap S$ and $\mathit{Gd}'(e) = \mathit{Gd}(e) \cap S$ for every location $q \in Q$ and transition $e \in \Delta$. We use $\mathcal{H}[F_1 \to F_2]$ to denote the hybrid system which is same as $\mathcal{H}$ except that the component F_1 ranging over Fl, Inv and Gd is replaced by F_2.

PROPOSITION 2. *Given a hybrid system $\mathcal{H}$ and $S \subseteq X$, an execution $\eta = (\sigma, \tau, \gamma)$ in $\mathit{Exec}(\mathcal{H})$ belongs to $\mathit{Exec}(\mathcal{H}[X \to S])$ if and only if $\sigma(t) \in S$ for every $t \in \mathit{Dom}(\sigma)$.*

3. STABILITY: LYAPUNOV AND ASYMPTOTIC

In this section, we define two classical notions of stability for hybrid systems, and state some general results about stability of hybrid systems.

We consider stability of the system with respect to an equilibrium point, which in our setting will be the origin.

Definition. $\mathbf{0}$ is an *equilibrium point* of a hybrid system $\mathcal{H}$ if any initial execution of $\mathcal{H}$ starting at $\mathbf{0}$ remains at $\mathbf{0}$.

Intuitively, Lyapunov stability captures the notion that an execution starting close to the equilibrium point remains close to it, and asymptotic stability, in addition, enforces converges to the equilibrium point.

Definition. A hybrid system $\mathcal{H}$ is said to be *Lyapunov stable*, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every initial execution $\eta \in \mathit{IExec}(\mathcal{H})$ with $\sigma(0) \in B_\delta(\mathbf{0})$, $\sigma(t) \in B_\epsilon(\mathbf{0})$ for every $t \in \mathit{Dom}(\sigma)$.

If $\mathcal{H}$ is Lyapunov stable, we use $\mathit{Lyap}(\mathcal{H}, \epsilon, \delta)$ to denote the fact that for every execution $\eta \in \mathit{IExec}(\mathcal{H})$ with $\sigma(0) \in B_\delta(\mathbf{0})$, $\sigma(t) \in B_\epsilon(\mathbf{0})$ for every $t \in \mathit{Dom}(\sigma)$.

In fact, we do not need to consider all possible values for ϵ in the definition of Lyapunov stability but only values in a small neighborhood around 0.

Definition. A hybrid system $\mathcal{H}$ is said to be *asymptotically stable*, if it is Lyapunov stable and there exists a $\delta > 0$ such that every complete initial execution $\eta \in \mathit{CExec}(\mathcal{H}) \cap \mathit{IExec}(\mathcal{H})$ with $\sigma(0) \in B_\delta(\mathbf{0})$ converges to $\mathbf{0}$, that is, for every $\epsilon > 0$, there exists a $T \in \mathit{Dom}(\sigma)$, such that $\sigma(t) \in B_\epsilon(\mathbf{0})$ for every $t \geq T$.

If $\mathcal{H}$ is asymptotically stable, we use $\mathit{Asymp}(\mathcal{H}, \delta)$ to denote the fact that every complete execution $\eta \in \mathit{CExec}(\mathcal{H})$ with $\sigma(0) \in B_\delta(\mathbf{0})$ converges to $\mathbf{0}$.

The next lemma says that to ensure Lyapunov stability it suffices to consider the restriction of $\mathcal{H}$ to a small enough neighborhood of $\mathbf{0}$. Let us use the shorthand $\mathcal{H}_\epsilon$ to represent $\mathcal{H}[X \to B_\epsilon(\mathbf{0})]$. The next lemma says that the stability of the system can be determined by examining its behavior in a small neighborhood around $\mathbf{0}$.

4. PLANAR RECTANGULAR SYSTEMS AND NORMAL FORM

In this section, we present some preliminary results which we will use later.

We consider a subclass of hybrid systems with rectangular flows and polyhedral invariants and guards, which we call rectangular hybrid systems. A hybrid system $\mathcal{H}$ with a partition $\mathcal{P}$ of X into convex polyhedral sets is a *rectangular hybrid system* if $Fl: Q \to Rect(X)$, $Inv: Q \to ConvPolyhed(X)$ and $Gd : Q \to ConvPolyhed(X)$, and for each $q \in Q$ and $e \in \Delta$, $Inv(q)$ and $Gd(e)$ are a finite union of the elements of $\mathcal{P}$. We will denote a rectangular system as a pair $(\mathcal{H}, \mathcal{P})$, however, when $\mathcal{P}$ is clear from the context we drop it from the notation. We use $Elements(\mathcal{H})$ to denote $Elements(\mathcal{P})$.

In this paper, we further focus on planar or 2-dimensional hybrid systems. We present certain structural properties of planar hybrid systems, and present a normal form for this class with respect to stability analysis.

A set S is said to be ϵ-*pie-shaped* if either

1. S is $\{\mathbf{0}\}$ or $\emptyset$,
2. S is a set $\{\alpha\mathbf{x} \mid 0 < \alpha < \epsilon\}$ for some $\mathbf{x}$ such that $|\mathbf{x}| = 1$, or
3. S is expressed as a set of points $(x, y) \in \mathbb{R}^2$ satisfying the constraints $a_1x + b_1y < 0$, $a_2x + b_2y < 0$ and $x^2 + y^2 < \epsilon$ for some $a_1, b_1, a_2, b_2 \in \mathbb{R}$.

We say that a planar hybrid system is ϵ-*pie-shaped* if for every location q and every transition e, $Inv(q)$ and $Gd(e)$ can be expressed as the union of a finite number of pie-shaped sets. The next lemma states that a rectangular hybrid system is a pie-shaped hybrid system when restricted to a small enough neighborhood around the origin.

LEMMA 1. *For a rectangular hybrid system $\mathcal{H}$, there exists $\epsilon > 0$, such that $\mathcal{H}_\epsilon$ is ϵ-pie-shaped.*

Given an ϵ-pie-shaped hybrid system, we construct a rectangular hybrid system by replacing each pie-shaped set by an unbounded conical set. Given an ϵ-pie-shaped set S, which is not $\{\mathbf{0}\}$, we define its infinite extension, denoted $InfExt(S)$, as follows. If $S = \{\alpha\mathbf{x} \mid 0 < \alpha < \epsilon\}$ for some $\mathbf{x}$ such that $|\mathbf{x}| = 1$, then $InfExt(S) = \{\alpha\mathbf{x} \mid \alpha \in \mathbb{R}_{\geq 0}/\{0\}\}$. If S is expressed as a set of points $(x, y) \in \mathbb{R}^2$ satisfying the constraints $a_1x + b_1y < 0$, $a_2x + b_2y < 0$ and $x^2 + y^2 < \epsilon$ for some $a_1, b_1, a_2, b_2 \in \mathbb{R}$, then $InfExt(S)$ is the set of points satisfying the constraints $a_1x + b_1y < 0$ and $a_2x + b_2y < 0$. Given an ϵ-pie-shaped system $\mathcal{H}$, we define $InfExt(\mathcal{H})$ to be the hybrid system obtained by replacing each invariant and guard given as the union of a finite set R of ϵ-pie-shaped sets, by the set which is the union of the sets $InfExt(S)$ for $S \in R$. Note that the result does not depend on the particular representation R of the invariant or guard.

Remark. Given an ϵ-pie-shaped hybrid system $\mathcal{H}$, the restriction to the ϵ-ball of its extension gives back $\mathcal{H}$, that is, $InfExt(\mathcal{H})_\epsilon = \mathcal{H}$.

The next proposition says that for a pie-shaped hybrid system, its stability is equivalent to that of its infinite extension.

PROPOSITION 3. *Let $\mathcal{H}$ be an ϵ-pie-shaped hybrid system. Then:*
$\mathcal{H}$ is Lyapunov (asymptotically) stable iff $InfExt(\mathcal{H})$ is Lyapunov (asymptotically) stable.

A planar rectangular hybrid system $\mathcal{H}$ is said to be in *normal form* if there exists an $\epsilon > 0$ such that $\mathcal{H}_\epsilon$ is ϵ-pie-shaped and $InfExt(\mathcal{H}_\epsilon)$ is $\mathcal{H}$.

PROPOSITION 4. *Given a planar rectangular hybrid system $\mathcal{H}$, there exists a planar rectangular hybrid system $\hat{\mathcal{H}}$ which is in normal form such that:*
$\mathcal{H}$ is Lyapunov (asymptotically) stable if and only if $\hat{\mathcal{H}}$ is Lyapunov (asymptotically) stable.

5. DECIDABILITY OF PLANAR RECTANGULAR HYBRID SYSTEMS

The algorithm for decidability consists of constructing a weighted graph and analyzing the graph for the absence of certain positive weight cycles. We need a primitive for constructing the graph, which is the maximum and minimum scaling of the state with respect to the start and end states of certain execution fragments. We formalize this primitive, present the construction of the graph and characterize the stability in terms of properties of this graph.

Elements of $\mathcal{H}$.

Let us fix a planar rectangular system $\mathcal{H}$ in normal form, as obtained by first taking its restriction to a small enough ϵ and then performing an infinite extension of its elements. The elements of $\mathcal{H}$, $Elements(\mathcal{H})$, are as follows:

1. The vertex $\{\mathbf{0}\}$. (We will abuse notation and represent $\{\mathbf{0}\}$ as just $\mathbf{0}$).
2. Lines $p_0, \cdots, p_{n-1}$, where p_i is an infinite ray starting from $\mathbf{0}$ but not including $\mathbf{0}$. (We will assume that the lines are numbered in the order in which they appear in the plane).
3. $R_{i,j}$, $0 \leq i, j \leq n-1$, $|i - j| = 1$, represents the open region between the lines p_i and p_j. (We assume that the addition of the indices i, j is modulo n).

Neighbors and elements between them.

We will define the concept of neighboring elements and elements between two neighboring elements as follows. The neighbors of a line p_i is $Neigh(p_i) = \{p_i, p_{i-1}, p_{i+1}, R_{i,i-1}, R_{i,i+1}, \mathbf{0}\}$. The elements between a line p_i and a neighboring element S of p_i, denoted $Between(p_i, S)$, is defined as follows:

1. If S is p_i, then $Between(p_i, S)$ is $\{p_i, R_{i,i+1}, R_{i-1,i}\}$.
2. If S is p_j and $|i - j| = 1$, then $Between(p_i, S)$ is $\{R_{i,j}\}$.
3. If S is $R_{i,j}$ for $|i - j| = 1$, then $Between(p_i, S)$ is $\{R_{i,j}\}$.
4. If S is $\mathbf{0}$, then $Between(p_i, S)$ is $\{p_i, R_{i,i+1}, R_{i-1,i}\}$.

Element and Location execution.

We will break up an execution into smaller fragments each of which either remains inside an element of $\mathcal{H}$ or inside a location of $\mathcal{H}$.

Definition. Given a line p_i, a neighbor S of p_i and an element $R \in Between(p_i, S)$, we say that an execution η of $\mathcal{H}$ is an *element execution* with respect to p_i, S and R if η is an execution from some point in p_i to some point in S such that $\sigma(t) \in R$ for every $0 < t < Size(Dom(\sigma))$.

We call η an element execution if it is an element execution with respect to some p_i, S and R.

Definition. Given a line p_i and a location q, we say that η is a *location execution* with respect to q and p_i if η is an execution from some point in p_i to some point in $R = p_i \cup R_{i,i-1} \cup R_{i,i+1}$ such that $\sigma(t) \in R$ for every $t \in \mathit{Dom}(\sigma)$, and $\gamma(i) = q$ for all $i \in \mathit{Dom}(\gamma)$.

We call η a location execution if it is a location execution with respect to some location q and some line p_i. Note that a location execution need not be an element execution because it could move in both the sets $R_{i,i-1}$ and $R_{i,i+1}$, where as an element execution requires it to stay in one of these sets.

Max and Min Scalings.

Next, we define four primitives which denote the maximum and minimum scaling between the starting and ending points of an element execution and a location execution.

Given a neighbor S of a line p_i and locations q_1, q_2 of $\mathcal{H}$, we define $\mathit{MaxSclE}(q_1, p_i, q_2, S)$ and $\mathit{MinSclE}(q_1, p_i, q_2, S)$ as follows. Let Σ_E denote the set of all executions η of $\mathcal{H}$ such that η is an element execution with respect to p_i, S, and R for some $R \in \mathit{Between}(p_i, S)$ and $\gamma(0) = q_1$ and $\gamma(|\mathit{Dom}(\gamma)|) = q_2$. Then:

- $\mathit{MaxSclE}(q_1, p_i, q_2, S) = \mathit{MinSclE}(q_1, p_i, q_2, S) = \perp$ if Σ_E is empty.
- $\mathit{MaxSclE}(q_1, p_i, q_2, S) = \sup_{\eta \in \Sigma_E} \mathit{Scaling}(\eta)$, and $\mathit{MinSclE}(q_1, p_i, q_2, S) = \inf_{\eta \in \Sigma_E} \mathit{Scaling}(\eta)$, otherwise.

Given a line p_i and a location q, let Σ_L denote the set of all location executions with respect to q and p_i.

- $\mathit{MaxSclL}(q, p_i) = \mathit{MinSclL}(q, p_i) = \perp$ if Σ_L is empty.
- Otherwise, $\mathit{MaxSclL}(q, p_i) = \sup_{\eta \in \Sigma_L} \mathit{Scaling}(\eta)$, and $\mathit{MinSclL}(q, p_i) = \inf_{\eta \in \Sigma_L} \mathit{Scaling}(\eta)$.

Construction of the graph.

Next, we present the construction of the graph. We construct the graph $G_{\mathcal{H}} = (V, E, V_0)$ with two weight functions W_{max} and W_{min} as follows:

1. $V = Q \times \mathit{Elements}(\mathcal{H})$.
2. $V_0 = \{(q, S) \in V \,|\, q \in Q_0, S \subseteq \mathit{Inv}(q)\}$.
3. $E = \{((q, p_i), (q', S)) \in V \times V \,|\, S \in \mathit{Neigh}(p_i), \mathit{MaxSclE}(q, p_i, q', S) \neq \perp\} \cup \{((q, p_i), (q, p_i)) \in V \times V \,|\, p_i \in P\}$.
4. $W_{max}((q, p_i), (q', S)) = \ln(\mathit{MaxSclE}(q, p_i, q', S))$ if $(q, p_i) \neq (q', S)$, $\mathit{max}(\ln(\mathit{MaxSclE}(q, p_i, q', S)), \ln(\mathit{MaxSclL}(q, p_i)))$ otherwise.
5. $W_{min}((q, p_i), (q', S)) = \ln(\mathit{MinSclE}(q, p_i, q', S))$ if $(q, p_i) \neq (q', S)$, $\mathit{max}(\ln(\mathit{MinSclE}(q, p_i, q', S)), \ln(\mathit{MinSclL}(q, p_i)))$ otherwise.

We will assume that the invariant associated with an initial location is a line. If the invariant is a region, then we can consider the lines of its closure (which are reachable) as the invariant for the purpose of the analysis, along with a test to check if the execution can blow up while remaining in the region.

Definition. Given an edge $e = ((q, p), (q', p'))$, we say that execution η *realizes* it, denoted $\eta \in \mathit{Realize}(e)$, if η is an element or location execution and satisfies $q = \gamma(0)$, $q' = \gamma(|\mathit{Dom}(\gamma)|)$, $\sigma(0) \in p$ and $\sigma(\mathit{Size}(\mathit{Dom}(\sigma))) \in p'$.

Definition. An execution η is said to *realize* an edge e of $G_{\mathcal{H}}$ with precision $\epsilon \geq 0$, denoted $\eta \in \mathit{Realize}_\epsilon(e)$, if it realizes e and $\mathit{Weight}(\eta) \in [W_{max}(e) - \epsilon, W_{max}(e)]$.

PROPOSITION 5. *Let $e = ((q_1, r_1), (q_2, r_2))$ be an edge of $G_{\mathcal{H}}$. Then:*

1. *Every execution η realizing an e satisfies $\mathit{Weight}(\eta) \leq W_{max}(e)$.*
2. *For every $\epsilon > 0$ and every $v_1 \in r_1$, there exists an η from (q_1, v_1) realizing e with precision ϵ.*

Characterization of Lyapunov Stability.

We need the notion of a q-cycle, negative cycle and an exploding node.

Definition. A cycle π of $G_{\mathcal{H}}$ is said to be a *q-cycle*, if for every $i \in \mathit{Dom}(\pi)$, $\pi(i) = (q, S)$ for some S.

Definition. A cycle π of $G_{\mathcal{H}}$ is a *negative* cycle if $W_{min}(\pi) < 0$.

Definition. A node $(q, \mathbf{0})$ of $G_{\mathcal{H}}$ is said to be *exploding* if there exists a location q' such that $\mathit{Trans}(q, q', \mathbf{0})$ and $\mathit{Fl}(q') \cap \mathit{Cone}(\mathit{Inv}(q'), \mathbf{0}) \neq \emptyset$, where $\mathit{Trans}(q_s, q_f, r_c)$ holds iff q_f is reachable from q_s by a series of transitions whose guards contain r_c.

LEMMA 2. *Any finite execution η of $\mathcal{H}$ which starts on some line p_i and does not reach $\mathbf{0}$ can be split into finitely many fragments $\eta_1, \cdots, \eta_n$ such that each η_i is either an element execution or a location execution.*

LEMMA 3. *If an execution η of $\mathcal{H}$ starting at some point in p_i at location q reaches $\mathbf{0}$ and then leaves $\mathbf{0}$, then in the graph $G_{\mathcal{H}}$, either there exists a path from the node (q, p_i) to an exploding node or there exists a simple negative q' cycle for some q' such that $(q', \{\mathbf{0}\})$ is an exploding node.*

THEOREM 1. *$\mathcal{H}$ is Lyapunov stable if and only if the following conditions hold:*

1. *There does not exist a simple cycle π in $G_{\mathcal{H}}$ such that $W_{max}(\pi) > 0$ and $\pi(0)$ is reachable from an initial node in V_0.*
2. *No edge (u, v) of $G_{\mathcal{H}}$ with $W_{max}((u, v)) = +\infty$ is reachable from an initial node.*
3. *For every exploding node $v = (q, \mathbf{0})$, v is not reachable from an initial node and there does not exist a simple negative q-cycle π such that $\pi(0)$ is reachable from an initial node.*

Characterization of Asymptotic Stability.

We need the notions of a locally stable region and a time bounded edge.

Definition. A region R in $\mathit{Elements}(\mathcal{H})$ is said to be *locally stable* with respect to a location q if every complete execution starting in (q, R) and remaining within R converges to $\mathbf{0}$.

PROPOSITION 6. *A region R in $\mathit{Elements}(\mathcal{H})$ is locally stable with respect to a location q iff $\mathbf{0} \notin \mathit{ConvHull}(S)$ and $S \setminus \mathit{Cone}(\mathit{Closure}(R), \mathbf{0}) = \emptyset$, where S is given by $\cup_{q'} \mathit{Fl}(q')$ such that q' is reachable from q by a series of transitions whose guards contain R.*

Further, if R is not locally stable with respect to q, then there exists a complete execution from every point in R starting at q which does not converge.

Definition. An edge $e = ((q_1, p_i), (q_2, p_i))$ with $W_{max}(e) = 0$ is said to be *bounded in time*, if for every $v_i \in p_i$, there exists a time bound $T \in \mathbb{R}_{\geq 0}$ such that all executions η from (q_1, v_i) realizing e with precision 1 satisfy $Size(Dom(\sigma)) \leq T$.

PROPOSITION 7. *Suppose $e = ((q_1, p_i), (q_2, p_i))$ is bounded in time and T is the bound on the time for a point $v_i \in p_i$, then for every $\epsilon > 0$, the time domain of any execution η realizing e from any point $\alpha v_i \in p_i$, $\alpha > 0$ with precision ϵ is bounded by $\epsilon\alpha T$.*

LEMMA 4. *Any complete execution η of $\mathcal{H}$ which starts on some line p_i and does not reach $\mathbf{0}$ can be split into either infinitely many fragments $\eta_1, \eta_2, \cdots$ such that each η_i is either an element execution or a location execution, or into finitely many fragments $\eta_1, \cdots, \eta_n, \eta'$, where η_i are element or location executions and η' is an execution which always remains in a particular region R.*

THEOREM 2. *$\mathcal{H}$ is asymptotically stable if and only if $\mathcal{H}$ is Lyapunov stable and the following conditions hold:*

1. *Every simple cycle π in $G_{\mathcal{H}}$ where $\pi(0)$ is reachable from an initial node in V_0 by a path with finite weight satisfies one of the following:*
   - *(a) $W_{max}(\pi) < 0$, or*
   - *(b) $W_{max}(\pi) = 0$, for every edge e of π, $W_{max}(e) = 0$, all the nodes of π correspond to a single p_i, and every edge of π is bounded in time.*
2. *Every node (q, R), where R is a region, which is reachable in $G_{\mathcal{H}}$ from an initial node, is locally stable.*

Algorithm for checking Theorem 1 and Theorem 2.

We will assume that all the constants appearing in the specification of the hybrid automaton are rational. The algorithm for checking Lyapunov and asymptotic stability involves constructing the graph $G_{\mathcal{H}}$ and checking the conditions in Theorem 1 and Theorem 2. Constructing the graph requires computation of the primitives *MaxSclE*, *MinSclE*, *MaxSclL* and *MinSclL*, which we will explain in the next section.

In terms of Lyapunov stability, we need to show that the primitives $Trans(q, q', r)$ can be computed and whether a node is exploding can be decided. One can compute the primitive $Trans(q, q', r)$ by first identifying the edges in the underlying graph of $\mathcal{H}$ which contain r and checking reachability using those edges. Containment, emptiness and intersection of sets expressed as first order logic formulas can be effectively computed, and so is the primitive $Cone(S, s)$ when S is a convex set. Hence, one can compute both primitives. Once the graph is constructed, the conditions involving existence of reachable simple cycles with positive and negative weights, reaching an edge with $+\infty$ weight or reaching an exploding node can be computed in time polynomial in the number of nodes of the graph. Hence, all the conditions in Theorem 1 can be effectively checked.

For checking asymptotic stability, the conditions on the graph can again be effectively checked. Determining local stability of a region involves checking for intersection of two sets, computing $Cone(S, s)$ for a convex polyhedral set S, and computing the convex hull of a finite number of convex sets, each of which can be effectively carried out using the decision procedures for first-order logic over $(\mathbb{R}, 0, +, <)$. It remains to show for deciding asymptotic stability that the property of an edge to be bounded in time can be decided, which we will show in the next section.

6. COMPUTING THE GRAPH ELEMENTS

In this section, we outline the procedure for computing Max and Min Scaling. Essentially, we perform a reachability computation and extract the min and max scaling from the set.

6.1 Reachability in a Convex Polyhedral Set

We compute the reachable points in a convex polyhedral set by executions which remain within the set.

Problem 1. *Let us fix a convex polyhedral set R which is open. Let $\mathcal{H}$ be a hybrid system whose invariants and guards are R. Let l be a line and r an element, both of which are in $Closure(R)$. Let q_1 and q_2 be locations in $\mathcal{H}$. Compute the set $Reach_{\mathcal{H}}(R, l, r, q_1, q_2)$, which is the set of all (v_1, v_2, t) such that $v_1 \in l$ and $v_2 \in r$ and there exists an incomplete execution η of $\mathcal{H}$ from (q_1, v_1) to (q_2, v_2) with $\sigma(Dom(\sigma)) = t$ and which remains within R, that is, $\sigma(t') \in R$ for $0 < t' < t$.*

The structure of the computation is similar to that in [14], but there are a few differences due to the more general dynamics we consider. Below are some series of reductions to a simpler problem that we need to solve.

Procedure 1. *Procedure for reducing to a hybrid system whose underlying graph is a strongly connected component.*

- *Compute $G/_{SCC}$, where $G = (Q, \Delta)$ is the underlying graph of $\mathcal{H}$.*
- *Note that $Reach_{\mathcal{H}}(R, l, r, q_1, q_2)$ is equivalent to computing the union of $Reach_{\mathcal{H}_\pi}(R, l, r, q_1, q_2)$ for every path $\pi = C_1 \cdots C_n \in G$, where $\mathcal{H}_\pi$ is the restriction of $\mathcal{H}$ to the locations in components of π, that is, $\cup_{1 \leq i \leq n} C_i$. Since the set of paths of G/SCC is finite, we can compute $Reach_{\mathcal{H}_\pi}(R, l, r, q_1, q_2)$ for each π.*
- *Let us fix a $\pi = D_1, \cdots, D_k$. We can further reduce the problem to reachability with respect to one strongly connected component as follows. We can assume w.l.o.g that $q_1 \in D_1$ and $q_2 \in D_k$.*
  - *First we compute the set of points (q, v') reached in R from (q_1, v) in $\mathcal{H}[Q \to D_1]$. Let us call this set S_1.*
  - *Next, for $i = 2, \cdots, k-1$, we iteratively compute S_i from S_{i-1} as follows. S_i is the set of points (q, v') reached in R starting from some point in S_{i-1} in the hybrid system $\mathcal{H}[Q \to D_i]$.*
  - *Finally, we compute the set of points (q, v') reached in r starting from some point in S_{k-1} in the hybrid system $\mathcal{H}[Q \to D_k]$.*

We will show that each of the sets S_i can be expressed as a formula of the first order logic over $(\mathbb{R}, 0, +, <)$, and hence is a finite union of convex polyhedral sets. From the above sketch of the algorithm, it follows that the primitive that we need to compute is the following.

Problem 2. *Let R be an open convex polyhedral set, and r_i, $i = 1, 2$ be a convex polyhedral sets each of which is a subset of some element of Closure(R). Let $\mathcal{H}$ be a hybrid system all of whose invariants and guards are R and its underlying graph is strongly connected. Let q_1 and q_2 be two locations in $\mathcal{H}$. We want to find the set of all triples (v_1, v_2, t) such that (q_2, v_2) is reached in r_2 from points (q_1, v_1) in r_1 by an executions of $\mathcal{H}$ whose domain is $[0, t]$.*

In the rest of the section, we show how to compute this. We focus on a single starting point $v \in r_1$. We prove a series of properties which reduce the problem to a simple computation. First, we consider a hybrid system $\mathcal{H}_1$ which is the same as $\mathcal{H}$ except that the invariants and guards are replaced by the whole Euclidean space, that is, $\mathbb{R}^2$. Note that solving Problem 2 for $\mathcal{H}_1$ gives us a set which is a superset of the set we would obtain for $\mathcal{H}$. We will show that they are equivalent modulo some exceptions.

The next property states that in $\mathcal{H}_1$ we only need to consider executions which are piecewise linear and whose derivative belong to a finite set of vectors. Let $\mathcal{H}_2$ be the hybrid system which is same as $\mathcal{H}_1$ except that for each location q, its flow $Fl_2(q) = Extremes(Fl_1(q))$.

PROPOSITION 8. *If v_2 is reachable in $\mathcal{H}_1[Q \to \{q\}]$ from v_1 by an execution η such that σ is differentiable everywhere in the interval $(0, T)$, then there exists an $a \in Fl_1(q)$ such that $v_2 = v_1 + aT$, where $T = Size(Dom(\sigma)))$.*

LEMMA 5. *v_2 is reachable from v_1 in the hybrid system $\mathcal{H}_1[Q \to \{q\}]$ by an execution in time T iff v_2 is reachable from v_1 in the hybrid system $\mathcal{H}_2[Q \to \{q\}]$ by an execution in time T.*

Remark. The above lemma states that reachability in a strongly connected component with $\mathbb{R}^2$ as invariants and guards can be interpreted as a single location with $\mathbb{R}^2$ as invariant and the union of the extremes of the flows of the locations in the strongly connected component as its flow. Note that the locations do not play a role since starting in any location one can take a series of transitions to any other location without any time elapse, due to the strongly connected nature of the graph.

Next we want to lift the results in the setting where invariants and guards are $\mathbb{R}^2$ to one where they are given by a convex polyhedral open set R. For that we use a result from [14] which shows that if some point $v_2 \in Closure(R)$ is reachable from $v_1 \in Closure(R)$ by following at most two flows, then v_2 can be reached from v_1 by switching finitely many times between the two flows while remaining within R as long as neither of them is a vertex. And if one of them is a vertex, then there is an additional test involving the flows of the locations which determines if it is reachable. We summarize the result from [14] below.

LEMMA 6. *Let R be a convex polyhedral open set. Suppose $v_2 \in Closure(R)$ is reachable from $v_1 \in Closure(R)$ using flows a_1 and a_2, that is, $v_2 = a_1t_1 + a_2t_2$ for some $t_1, t_2 \geq 0$. Let us associate a set S_i with the vertices v_i, $i = 1, 2$ as follows. The set associated with v_i is empty if v_i is not a vertex. Otherwise, let S_i be a singleton set with a non-zero vector v such that there exists $t > 0$ with $v_i + tv \in R$ if $i = 1$ and $v_i - tv \in R$ otherwise. Then:*

- *There exists a piecewise continuous function $f : [0, T] \to \mathbb{R}^2$ such that $f(0) = v_1$, $f(T) = v_2$, $T = t_1 + t_2$ and the derivative in each piece belongs to $\{a_1, a_2\} \cup S_1 \cup S_2$ such that $f(t) \in R$ for every $t \in (0, T)$.*

In fact, we can generalize Lemma 6 to any number of flows. This tells us that reachability in $\mathcal{H}_2$ is almost equivalent to reachability in $\mathcal{H}$, except for the condition that we need to be able to enter R from our initial state and exit to the final state. We formalize this fact below.

THEOREM 3. *Let R be an open convex polyhedral set. Let r_1 and r_2 be elements of Closure(R). Let $\mathcal{H}$ be a hybrid system whose underlying graph is strongly connected and all of whose guards and invariants are R. Let $t > 0$. The following are equivalent:*

1. *A point v_2 in r_2 at location q_2 is reachable from a point v_1 in r_1 at location q_1 by an execution η of $\mathcal{H}$ with $Size(Dom(\sigma)) = t$ which remains within R in the interval $(0, t)$.*

2. *Let $\cup_{q \in Q} Extremes(Fl(q)) = \{a_1, \cdots, a_n\}$. There exist times $t_0, t_1, \cdots, t_n, t_{n+1} \geq 0$ with $t_0, t_n > 0$ and $\sum_{i=0}^{n+1} t_i = t$, and flow $a_0 \in Fl(q_1')$ and $a_{n+1} \in Fl(q_2')$ for some location q_1' and q_2' such that $Trans(q_1, q_1', r_1)$ and $Trans(q_2', q_2, r_2)$ holds, $v_2 = v_1 + a_0t_0 + \cdots + a_nt_n + a_{n+1}t_n$ and $v_1 + a_0t', v_2 - a_{n+1}t'' \in R$ for every $t' \in (0, t_0)$ and $t'' \in (0, t_{n+1})$, where $Trans(q_s, q_f, r_c)$ holds iff q_f is reachable from q_s by a series of transitions whose guards contain r_c.*

The above theorem summarizes the algorithm for solving Problem 2. One can write a first order logic formula over $(\mathbb{R}, 0, +, <)$ with free variables for v_1, v_2 and t. Let us call this formula $Reach_{\mathcal{H},R,r_1,r_2,q_1,q_2}(v_1, v_2, t)$. This formula can be used in Procedure 1 to iteratively compute a formula for each $\pi \in G_{SCC}$, and then for $\mathcal{H}$ itself.

6.2 Computing the primitives

Computing reachability by element executions.

Given p_i, a neighbor S of p_i, an element $R \in Between(p_i, S)$ and locations q_1 and q_2 of the hybrid system $\mathcal{H}$, we want to compute the set of points reached in S at location q_2 from the points in p_i at location q_1 by executions which remain within R except at the end-points. When R is an open set, it is equivalent to computing $Reach_{\mathcal{H}}(R, l, r, q_1, q_2)$. When R is a line or a point, we essentially follow Procedure 1 except that the analogue of Problem 2 can be solved by much simpler tests. For example, if R is a line, then essentially, $p_i = R$ and $S = p_i$ or $\mathbf{0}$. The set of points computed is the union of a subset of elements from the set $\{U, L, \mathbf{0}\}$, where U is the set of all points "above" v_i in p_i and L is the set of all points "below" v_i in p_i. And the inclusion of each of the set U, L and $\mathbf{0}$ can be tested easily. For example, U is included iff there exists a vector along p_i in the positive direction in one of the locations reachable from the given location.

Computing reachability by location executions.

Given p_i and q, we want to compute the set of all points reached by finite executions starting from some point in p_i at location q which remain within $R = p_i \cup R_{i,i-1} \cup R_{i,i+1}$ and in location q all the time. Note that R is an open set and p_i is a convex subset of R. So, the problem reduces to Problem 2, with $\mathcal{H}[Q \to \{q\}]$.

Determining edges which are bounded in time.

It is straightforward to write a first order logic formula over $(\mathbb{R}, 0, <, +)$ which determines if an edge $((q_1, p_i), (q_2, p_i))$ with $W_{max}(e) = 0$ is bounded in time, since we can not only compute which point is reachable from which but also the time taken by the execution, as given by the predicate $Reach_{\mathcal{H},R,r_1,r_2,q_1,q_2}(v_1, v_2, t)$.

Max and Min Scaling.

First, we prove a property which says that the set of scalings associated with any point on a line p_i are the same.

PROPOSITION 9. *Let p_i be a line, S a neighbor of p_i and $R \in Between(p_i, S)$. Let q be a location. Then for every $v_1, v_2 \in p_i$,*

1. *if there exists an element execution η_1 from v_1 with respect to p_i, S and R, then there exists an element execution η_2 with respect to p_i, S and R with $Scaling(\eta_1) = Scaling(\eta_2)$, and*
2. *if there exists a location execution η_1 from v_1 with respect to q and p_i, then there exists a location execution η_2 with respect to q and p_i such that $Scaling(\eta_1) = Scaling(\eta_2)$.*

The above lemma states that $MaxSclE, MinSclE, MaxSclL$ and $MinSclL$ can be computed by focusing on a point v_i on the line p_i. To compute $MaxSclE$ and $MinSclE$, we compute the reachability by element execution with a fixed point v_i as the initial state and compute the maximum and minimum scaling from the set (by constructing a formula with one free variable for max or min using the predicate *Reach* and performing quantifier elimination). Similarly, to compute $MaxSclL$ and $MinSclL$, we compute reachable sets with respect to location executions and find the maximum and minimum scaling.

7. UNDECIDABILITY OF STABILITY OF PCD IN 5 DIMENSIONS

In this section, we show that Lyapunov and asymptotic stability are undecidable for rectangular hybrid systems. In particular, we show that for a subclass of rectangular hybrid systems called piecewise constant derivative systems, Lyapunov and asymptotic stability are undecidable in 5 dimensions.

Piecewise constant derivatives is a subclass of rectangular hybrid systems with the additional constraint that the invariants in distinct locations are disjoint and the guard associated with a transition is the common boundary, if any, of the closure of the invariants associated with the two locations in the transition. Further, and the flow associated with every location is a singleton set. We will use a simpler notation to represent such systems as defined below. A *piecewise constant derivative* (*PCD*) $\mathcal{H}$ of dimension d is a triple $(\mathcal{P}, \varphi, P_0)$, where $\mathcal{P}$ is a partition of $\mathbb{R}^d$ into convex polyhedral sets, $\varphi : \mathcal{P} \to \mathbb{R}^d$ associates a vector with each region of the partition, and P_0 is an initial region.

In [1], Asarin, Maler and Pnueli show that a pushdown automaton with 2 stacks (2-*PDA*) can be simulated by a 3-dimensional *PCD*, such that the control state reachability problem of 2-*PDA* is equivalent to "point-to-point" reachability of 3-dimensional *PCD*, where point-to-point reachability is the problem of deciding, given points v_0 and v_f in the state space of the *PCD*, if there exists an execution from v_0 to v_f. Since the control state reachability problem for 2-*PDA* is undecidable, the point-to-point reachability for 3-dimensional *PCD* is undecidable as well. Below we highlight some of the properties of the *PCD*s which are output by the reduction.

1. The initial partition P_0 is given by the initial vector $\{v_0\}$.
2. Since *PCD* are deterministic, there exists a unique maximal execution starting from the initial state v_0. If the 2-PDA is halting, then the unique maximal initial execution of the *PCD* is incomplete and reaches v_f in finite time, otherwise it has a complete execution which does not reach v_f.
3. The reachable state space of the *PCD* is bounded, that is, the set of points reached by initial executions is contained within a bounded region.

Below we summarize the undecidability result for *PCD* in [1].

PROPOSITION 10 ([1]). *The problem of point-to-point reachability from a point v_0 to a point v_f for the class of 3-dimensional PCD, whose reachable state space is bounded and in which the unique maximal trajectory from v_0 is either incomplete and reaches v_f, or is complete and does not reach v_f, is undecidable.*

We reduce the above problem to the problem of deciding Lyapunov and asymptotic stability. We take a *PCD* $\mathcal{H}_3$ of the above form and construct a 5-dimensional *PCD* such that point-to-point reachability of $\mathcal{H}_3$ is equivalent to Lyapunov (asymptotic) stability of $\mathcal{H}_5$. This exhibits the undecidability of Lyapunov and asymptotic stability for *PCD* in 5 dimensions.

We explain the construction from $\mathcal{H}_3$ to $\mathcal{H}_5$. First, we construct a 4-dimensional *PCD* $\mathcal{H}_4$. Intuitively, $\mathcal{H}_4$ is obtained by stacking uncountable many copies of a scaled version of $\mathcal{H}_3$ in a fourth dimension. That is, for each value α of the fourth dimension, we place at α a copy of $\mathcal{H}_3$ scaled by a factor of α. Concretely, we replace each element P of the partition $\mathcal{P}_3$ associated with $\mathcal{H}_3$, by a 4 dimensional polyhedral set as follows. We can assume without loss of generality that P is expressed as the conjunction of finitely many linear constraints of the form $ax+by+cz \sim d$ for some $a, b, c, d \in \mathbb{R}$ and $\sim \in \{<, \leq\}$. We obtain the 4 dimensional *PCD* P' by replacing each constraint $ax + by + cz \sim d$ in the variables x, y, z by the constraint $ax + by + cz \sim hd$ in the variables x, y, z and h. The vector associated with the element P' in $\mathcal{H}_4$ is the extension of the vector associated with P by the value 0 in the fourth dimension.

Remark. Observe that the only points reachable from each other in $\mathcal{H}_4$ are those whose fourth components (corresponding to the fourth dimension) have the same value. Note that if $y \in \mathbb{R}^3$ is reached from $x \in \mathbb{R}^3$ in time t in $\mathcal{H}$, then for every h, $(hy, h) \in \mathbb{R}^4$ is reached in time th from $(hx, h) \in \mathbb{R}^4$ in $\mathcal{H}_4$.

The last dimension we add corresponds to time. We add a variable which is a clock, that is, evolves at rate 1. More precisely, every partition in $\mathcal{H}_5$ is obtained by interpreting the constraints of the corresponding partition in $\mathcal{H}_4$ as a constraint over 5 variables, where the coefficient associated with the 5-th variable is always 0. The vector associated with a partition is extended in the 5-th dimension by the value 1. The initial partition of $\mathcal{H}_5$ is the set of points $\{(v_0, h, 0) \mid h \geq 0\}$.

Next, we present a proof of the correctness of the reduction.

THEOREM 4. *The following are equivalent:*

1. *v_f is reachable from v_0 in $\mathcal{H}_3$.*
2. *$\mathcal{H}_5$ is Lyapunov stable.*
3. *$\mathcal{H}_5$ is asymptotically stable.*

PROOF. (1) to (2): We show that if v_f is reachable from v_0, then $\mathcal{H}_5$ is Lyapunov stable. Let us fix an $\epsilon > 0$. Since the reachable state space of $\mathcal{H}_3$ is bounded, assume that it is contained in $B_d(\mathbf{0})$ for some $d > 0$. Then we know that all the points starting with value h for the 4-th variable in $\mathcal{H}_4$ are contained within a dh-ball around $(\mathbf{0}, h)$, or equivalently in a $\sqrt{d^2h^2 + h^2}$-ball around the origin. Since every execution of $\mathcal{H}_3$ is a prefix of the maximal initial execution η which is incomplete, the maximum time elapsed to reach v_f from v_0 is bounded by the domain of σ, say, T. Observe that any initial execution of $\mathcal{H}_4$ starting at (v_0, h) and ending at (v_f, h) takes time at most Th, since the flows evolve at the same rate as in $\mathcal{H}_3$ but the regions are scaled by a factor of h. Therefore, the execution starting from $(v_0, h, 0)$ in $\mathcal{H}_5$ at any time is in a $\sqrt{d^2h^2 + h^2 + T^2h^2}$-ball around $\mathbf{0}$. Choosing $\delta < \epsilon/\sqrt{d^2 + T^2 + 1}$, ensures that every point starting at $(v_0, \delta', 0)$ for some $\delta' \leq \delta$ remains within the ϵ-ball. But all the points starting within a δ-ball are a subset of the points in $\{(v_0, \delta', 0) \mid \delta' \leq \delta\}$. Hence, every point starting within the δ-ball around the origin remains within an ϵ-ball around the origin. Therefore, $\mathcal{H}_5$ is Lyapunov stable.

(1) to (3): (1) implies that $\mathcal{H}_5$ is Lyapunov stable. But then the system does not have any complete executions, therefore the system is trivially asymptotically stable.

(2) to (1): Next, we show that if $\mathcal{H}_5$ is Lyapunov stable, then v_f is reachable from v_0 in $\mathcal{H}_3$. Suppose not. Then the execution η of $\mathcal{H}_3$ is complete. So the corresponding execution starting at any initial point in $\mathcal{H}_5$ evolves unboundedly in the 5-th dimension, which is time. This implies that $\mathcal{H}_5$ is not Lyapunov stable, a contradiction to our assumption.

(3) to (2): If follows from the definition of asymptotic stability. This completes the proof. $\square$

8. CONCLUSIONS

We proved two main results. First we showed that checking (Lyapunov and asymptotic) stability of planar rectangular switched hybrid systems is decidable. Second, we showed that the same problem is undecidable for systems in 5 dimensions. There are a number of questions left open by our investigations. First, there are a couple of questions left unresolved that are intimately tied to our paper: can we reduce the dimensionality gap between our decidability and undecidability results for rectangular switched systems; and, is reachability decidable for planar rectangular switched systems. Next, more generally, one would like to study the stability problem for other classes of hybrid systems, and in particular, identify other decidable subclasses. Finally, one would like to understand the formal relationship between reachability and stability from a computational standpoint: can one of these problems be reduced to the other in general or are these problems computationally incomparable.

9. ACKNOWLEDGEMENT

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