Computer-aided cryptographic proofs

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Modern cryptography

Shannon ’49

- Mathematical proof of security
- Perfect secrecy is impossible

Diffie & Hellman ’76

Goldwasser & Micali’82

Yao’82

- Computational security
- Asymptotic guarantees

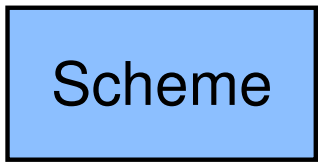
PPT adversary has negligible advantage

Bellare & Rogaway ’94

- Concrete bounds

Aversary advantage to win in time t is $\leq p$

Reductionist proof



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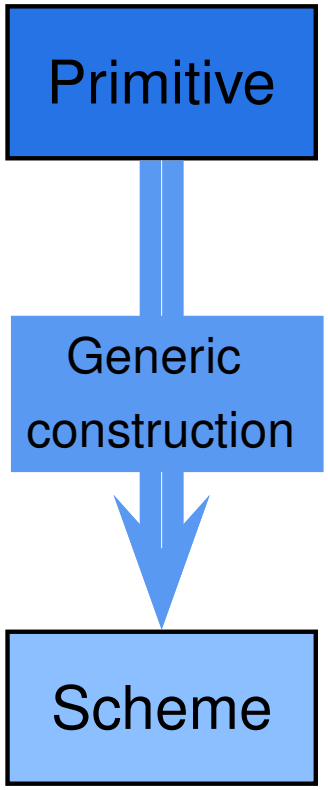

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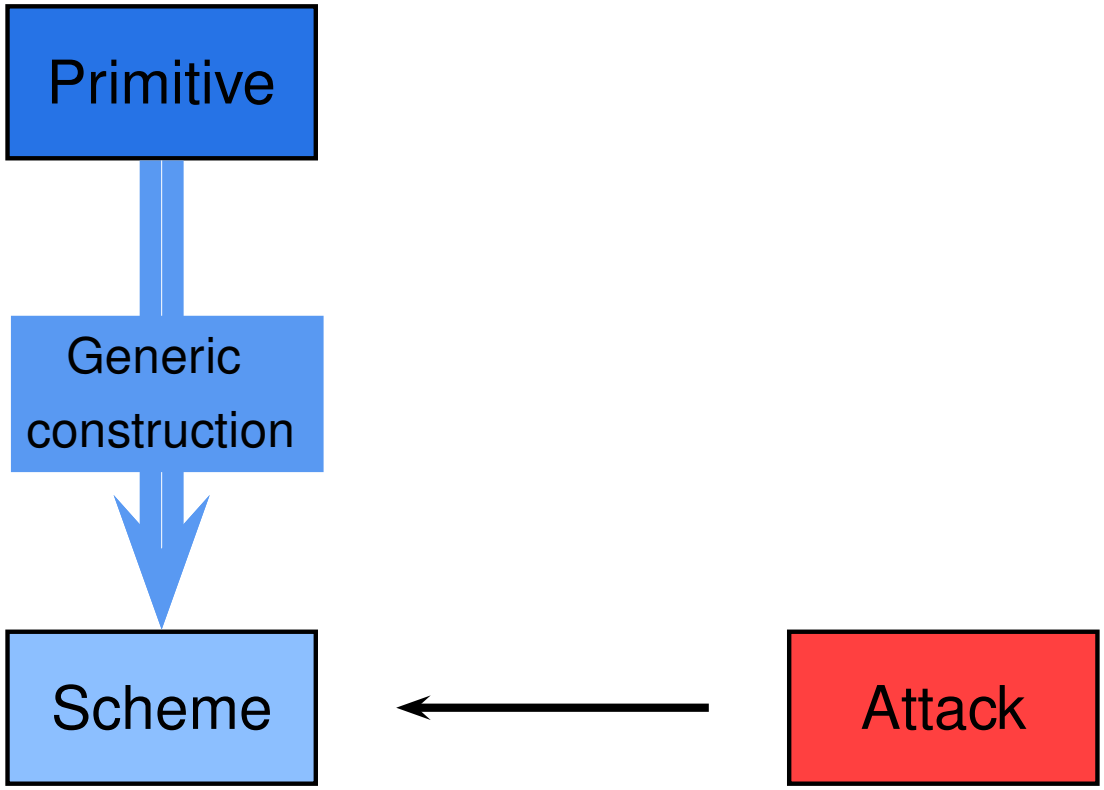

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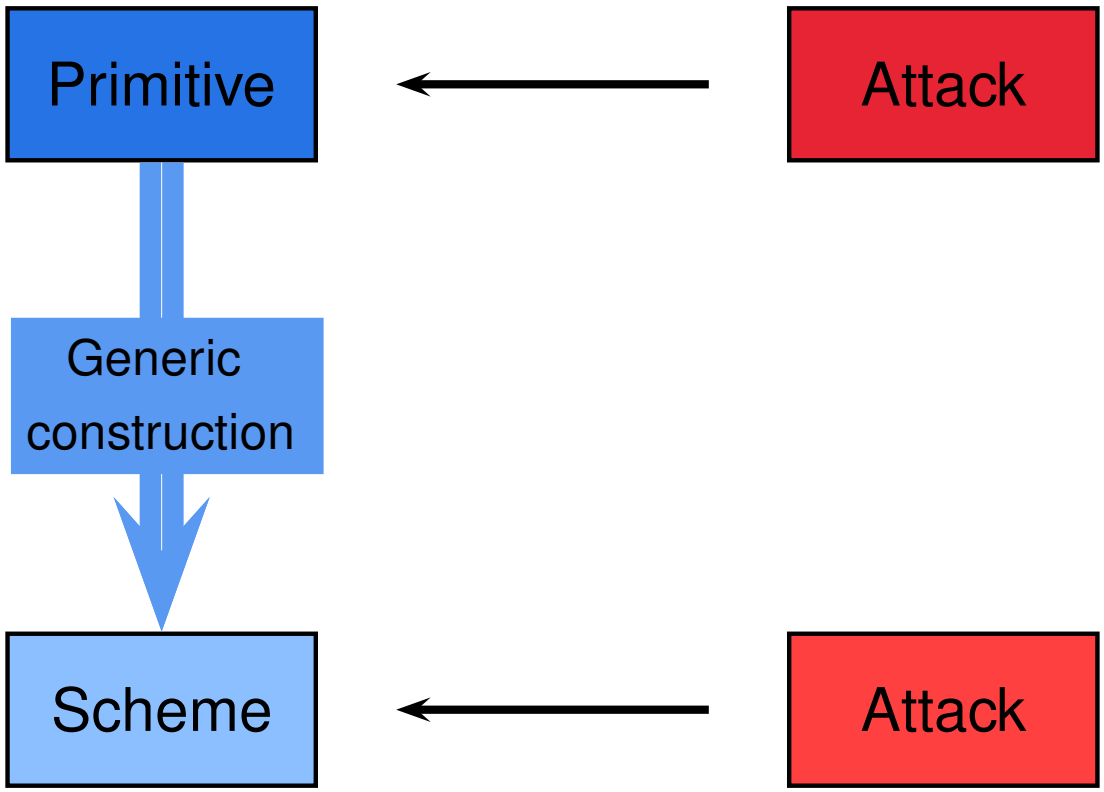

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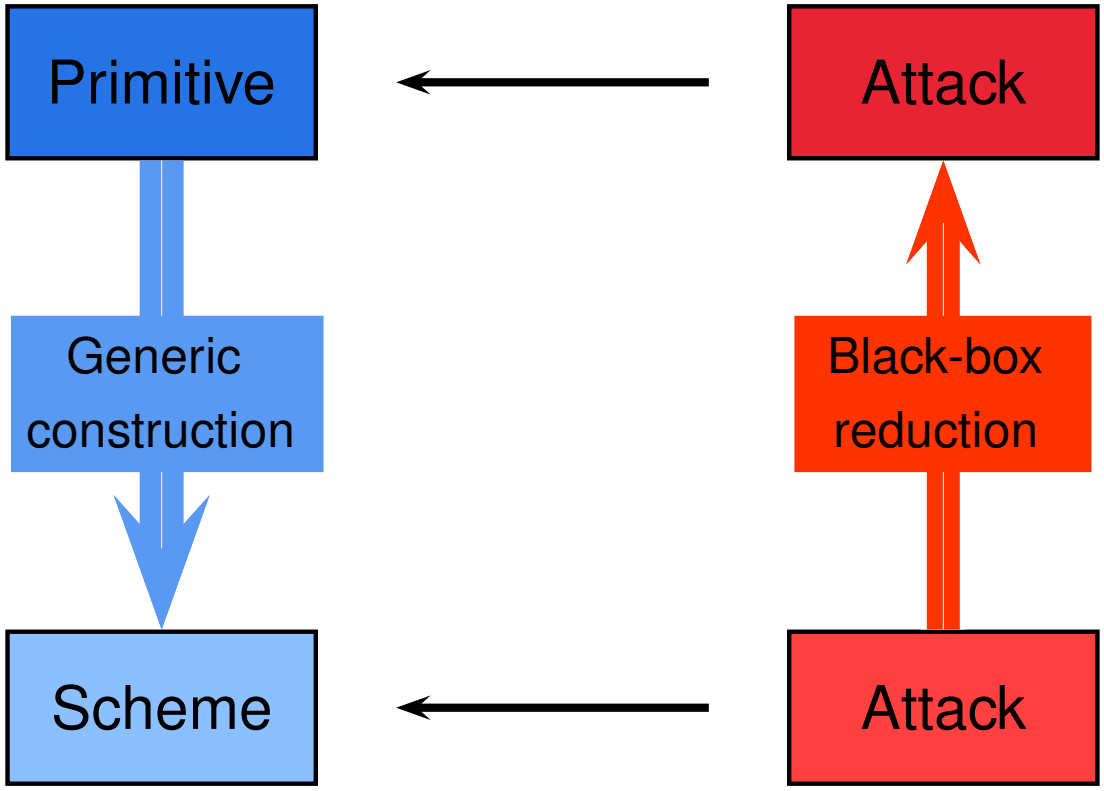


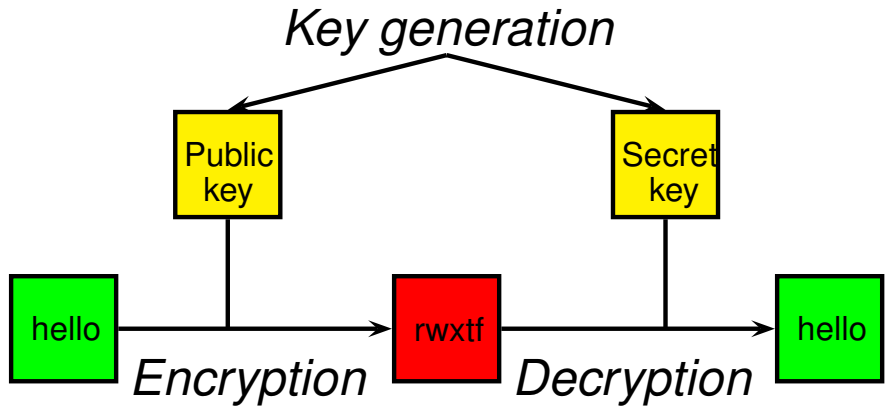
Public-key encryption

Algorithms $(\mathcal{K}, \mathcal{E}_{pk}, \mathcal{D}_{sk})$

- $\mathcal{E}$ probabilistic
- $\mathcal{D}$ deterministic and partial

If (sk, pk) is a valid key pair,

$$\mathcal{D}_{sk}(\mathcal{E}_{pk}(m)) = m$$



Indistinguishability

Game IND-CPA($\mathcal{A}$)
$(sk, pk) \leftarrow \mathcal{K}()$;
$(m_0, m_1) \leftarrow \mathcal{A}_1(pk)$;
$b \xleftarrow{\$} \{0,1\}$;
$c^\star \leftarrow \mathcal{E}_{pk}(m_b)$;
$b' \leftarrow \mathcal{A}_2(c^\star)$;
return $(b' = b)$

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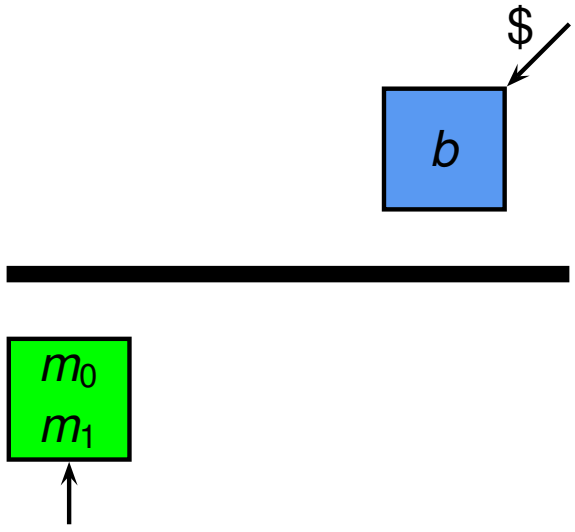

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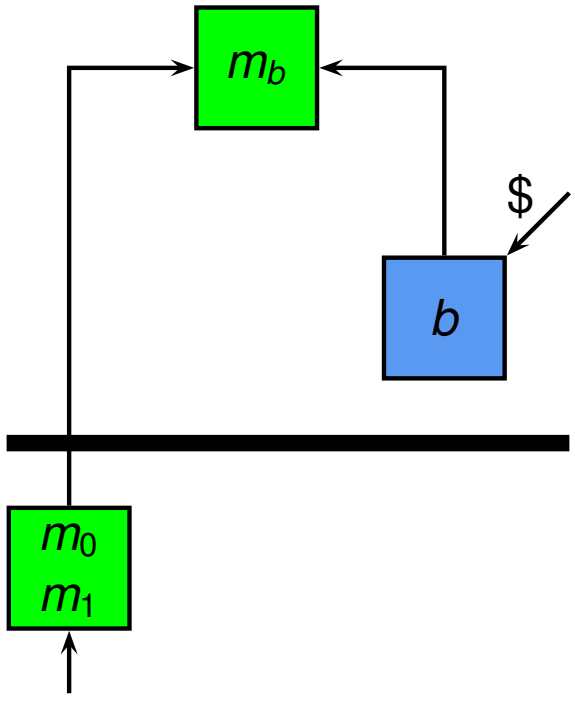

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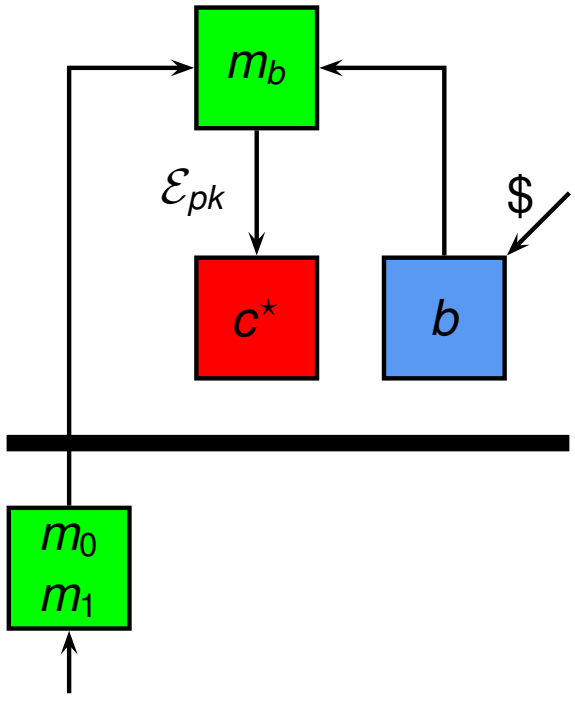

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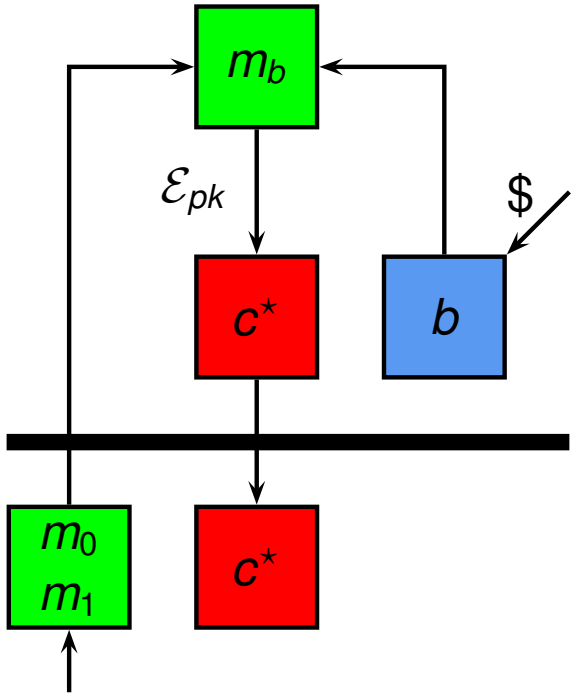

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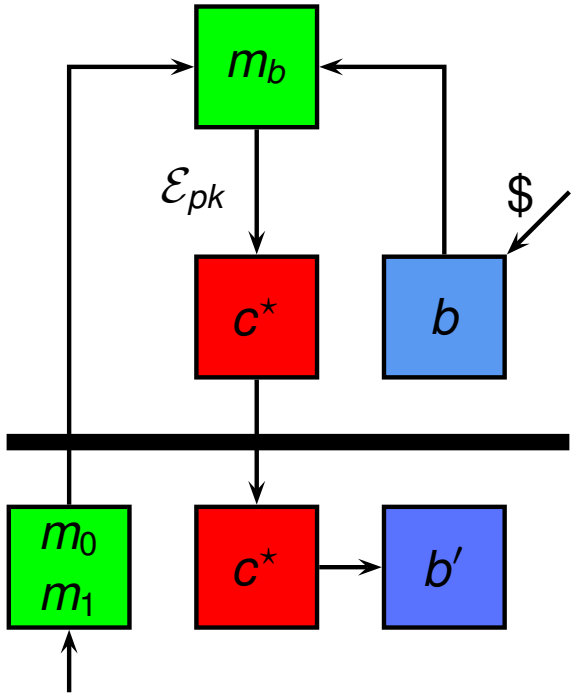

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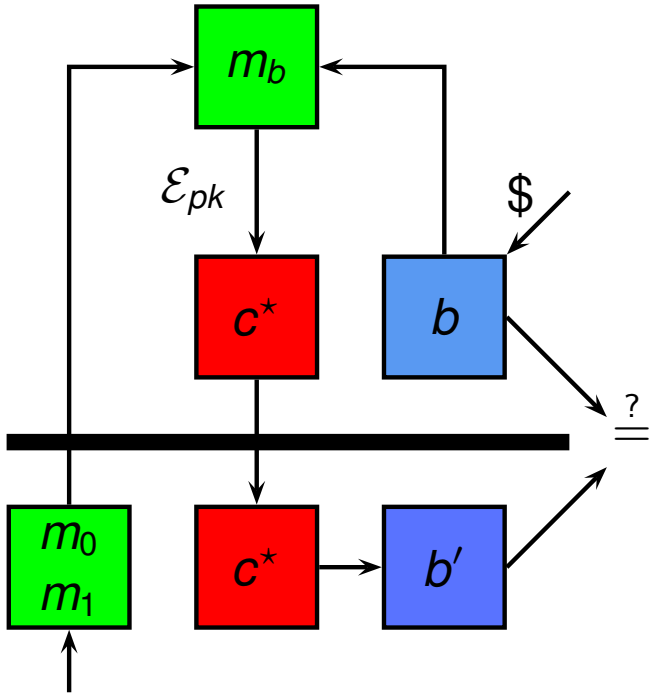

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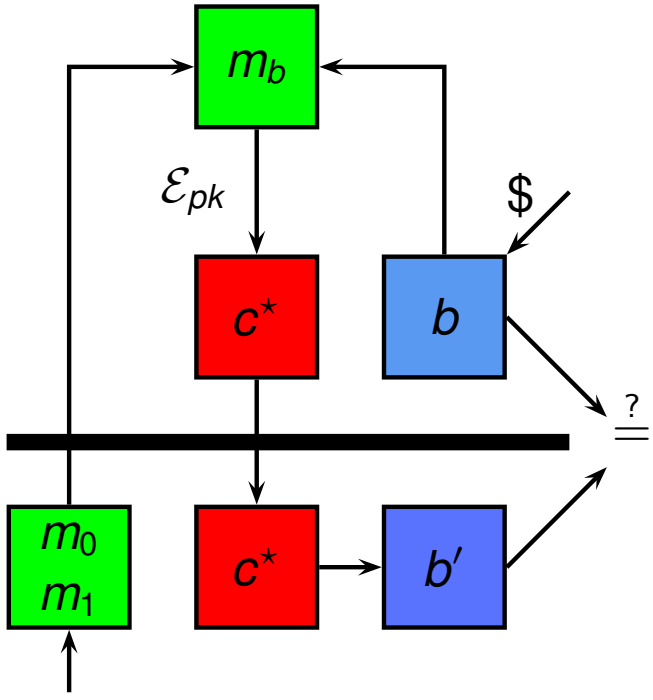


$$\Pr_{\text{IND-CPA}(\mathcal{A})}\left[b' = b\right] - \frac{1}{2} \quad \text{small}$$

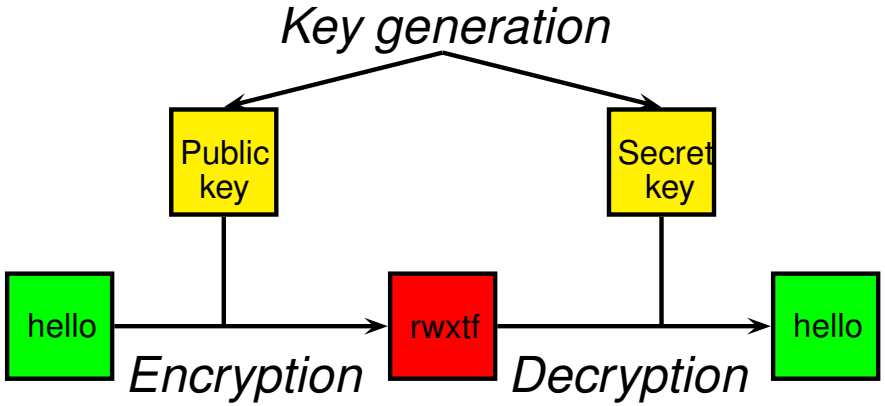
One-way trapdoor permutations

Algorithms $(\mathcal{K}, \mathsf{f}_{pk}, \mathsf{f}_{sk}^{-1})$

- f_{pk} and f_{sk}^{-1} deterministic

If (sk, pk) is a valid key pair,

$$\mathsf{f}_{sk}^{-1}(\mathsf{f}_{pk}(m)) = m$$



One-way trapdoor permutations

Game $\mathrm{OW}(\mathcal{I})$
$(sk, pk) \leftarrow \mathcal{K}()$;
$y \xleftarrow{\$} \{0,1\}^n$;
$x^\star \leftarrow \mathsf{f}_{pk}(y)$;
$y' \leftarrow \mathcal{I}(x^\star)$;
return $(y' = y)$

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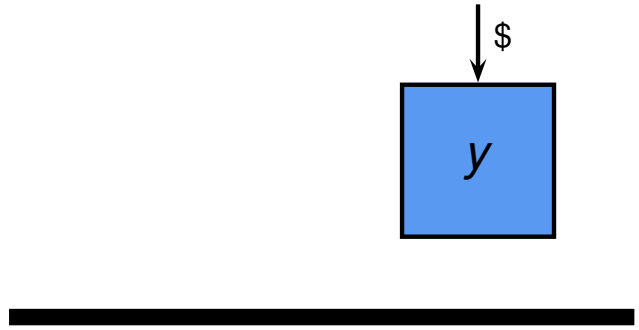

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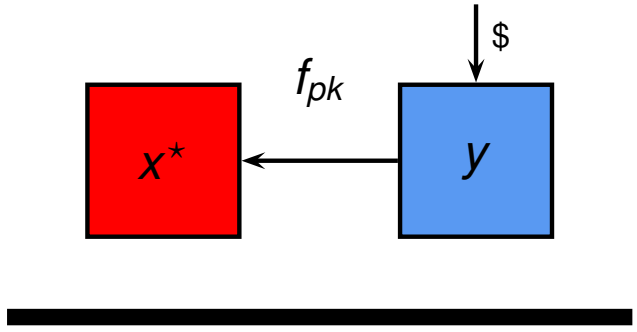

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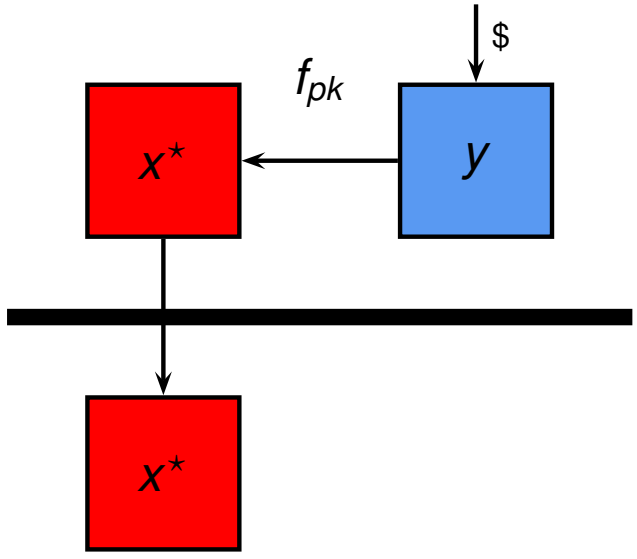

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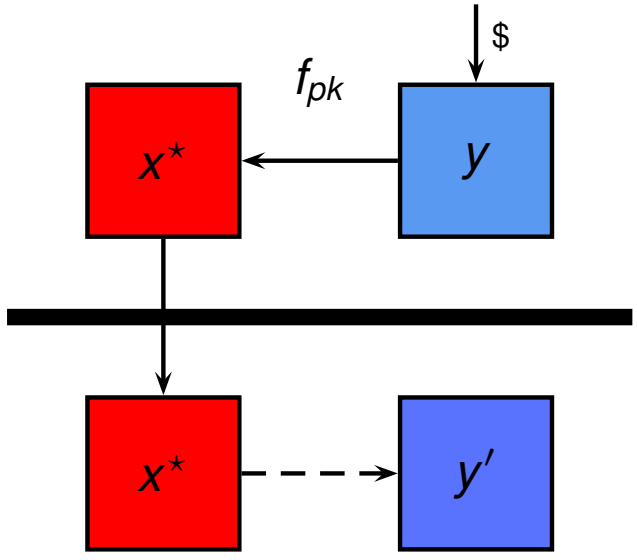

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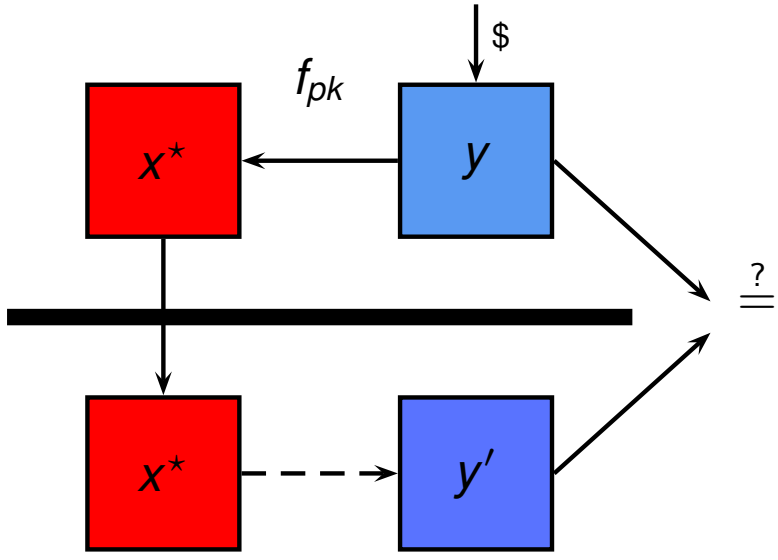

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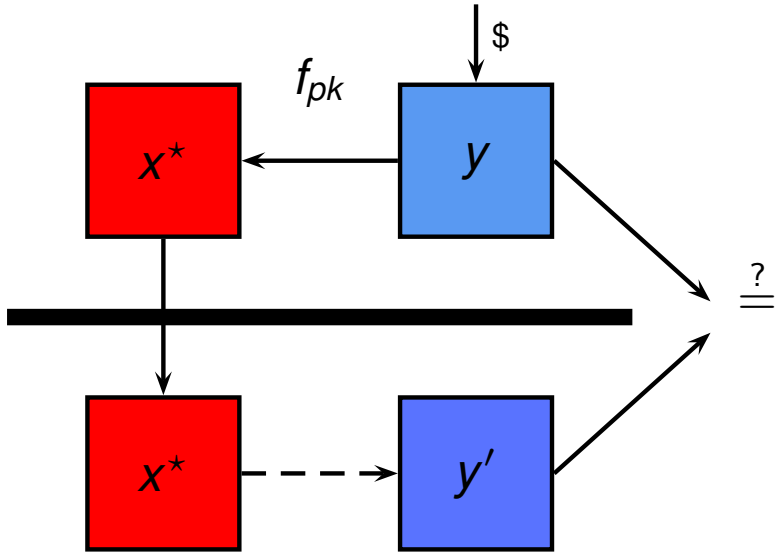


$$\Pr_{\mathrm{OW}(\mathcal{I})}\left[y' = y\right] \quad \text{small}$$

Random oracles

Oracle $H(x)$:
if $x \notin L$ then
  $r \xleftarrow{\$} \{0,1\}^k$;
  $L \leftarrow (x, r) :: L$;
return $L[x]$;

- Idealized model of hash function
- Allows practical schemes
- Not realizable

Optimal Asymmetric Encryption Padding

Encryption $\mathcal{E}_{\mathrm{OAEP}(pk)}(m)$:
$r \xleftarrow{\$} \{0,1\}^{k_0}$;
$s \leftarrow G(r) \oplus (m \| 0^{k_1})$;
$t \leftarrow H(s) \oplus r$;
return $\mathsf{f}_{pk}(s \| t)$

Decryption $\mathcal{D}_{\mathrm{OAEP}(sk)}(c)$:
$(s,t) \leftarrow \mathsf{f}_{sk}^{-1}(c)$;
$r \leftarrow t \oplus H(s)$;
if $([s \oplus G(r)]_{k_1} = 0^{k_1})$
then $\{m \leftarrow [s \oplus G(r)]^{k}; \}$
else $\{m \leftarrow \perp; \}$
return m

$\oplus$ exclusive or $\|$ concatenation $[\cdot]$ projection 0 zero bitstring

OAEP: provable security

Game IND-CCA($\mathcal{A}$)
$(sk, pk) \leftarrow \mathcal{K}()$;
$(m_0, m_1) \leftarrow \mathcal{A}_1(pk)$;
$b \xleftarrow{\$} \{0,1\}$;
$c^\star \leftarrow \mathcal{E}_{pk}(m_b)$;
$b' \leftarrow \mathcal{A}_2(c^\star)$;
return $(b' = b)$

Game SPDOW($\mathcal{I}$)
$(sk, pk) \leftarrow \mathcal{K}()$;
$y \xleftarrow{\$} \{0,1\}^{k_2}$; $z \xleftarrow{\$} \{0,1\}^{k_3}$;
$x^\star \leftarrow \mathsf{f}_{pk}(y \,\|\, z)$;
$Y' \leftarrow \mathcal{I}(x^\star)$;
return $(y \in Y')$

FOR ALL IND-CCA adversary $\mathcal{A}$ against $(\mathcal{K}, \mathcal{E}_{\mathrm{OAEP}}, \mathcal{D}_{\mathrm{OAEP}})$,
THERE EXISTS a SPDOW adversary $\mathcal{I}$ against $(\mathcal{K}, \mathsf{f}, \mathsf{f}^{-1})$ st

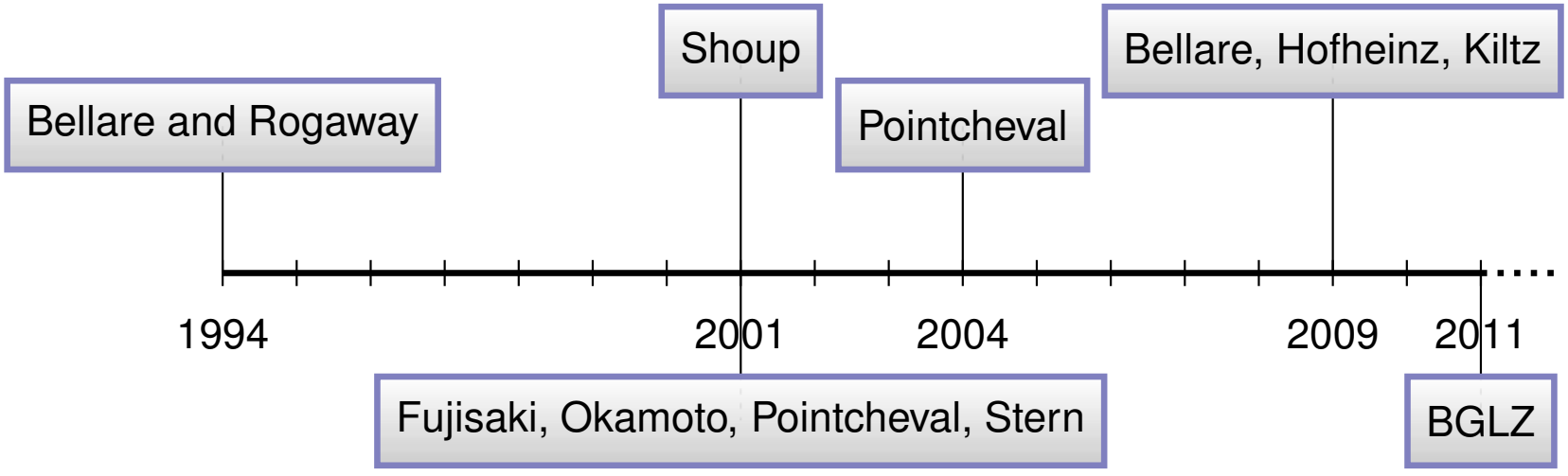
$$\left|\Pr_{\text{IND-CCA}(\mathcal{A})}[b' = b] - \tfrac{1}{2}\right| \leq$$

$$\Pr_{\text{SPDOW}(\mathcal{I})}[y \in Y'] + \frac{3q_D q_G + q_D^2 + 4q_D + q_G}{2^{k_0}} + \frac{2q_D}{2^{k_1}}$$

and

$$t_{\mathcal{I}} \leq t_{\mathcal{A}} + q_D\, q_G\, q_H\, T_{\mathsf{f}}$$

OAEP: provable security



1994 Purported proof of chosen-ciphertext security

2001 1994 proof gives weaker security; desired security holds

- for a modified scheme
- under stronger assumptions

2004 Filled gaps in 2001 proof

2009 Security definition needs to be clarified

2011 Fills gaps in 2004 proof

Example: Bellare and Rogaway 1993 encryption

Game IND-CPA($\mathcal{A}$) :
$(sk, pk) \leftarrow \mathcal{K}(\ )$;
$(m_0, m_1) \leftarrow \mathcal{A}_1(pk)$;
$b \xleftarrow{\$} \{0,1\}$;
$c^\star \leftarrow \mathcal{E}_{pk}(m_b)$;
$b' \leftarrow \mathcal{A}_2(c^\star)$;
return $(b' = b)$

Encryption $\mathcal{E}_{pk}(m)$:
$r \xleftarrow{\$} \{0,1\}^\ell$;
$s \leftarrow H(r) \oplus m$;
$y \leftarrow f_{pk}(r) \,\|\, s$;
return y

For every IND-CPA adversary $\mathcal{A}$, there exists an inverter $\mathcal{I}$ st

$$\Pr_{\text{IND-CPA}(\mathcal{A})}\left[b' = b\right] - \frac{1}{2} \leq \Pr_{\text{OW}(\mathcal{I})}\left[y' = y\right]$$

Proof

Game hopping technique

Game INDCPA :
$(sk, pk) \leftarrow \mathcal{K}();$
$(m_0, m_1) \leftarrow \mathcal{A}_1(pk);$
$b \xleftarrow{\$} \{0,1\};$
$c^\star \leftarrow \mathcal{E}_{pk}(m_b);$
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Game G :
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Encryption $\mathcal{E}_{pk}(m)$:
$r \xleftarrow{\$} \{0,1\}^\ell;$
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Game G′ :
$(sk, pk) \leftarrow \mathcal{K}();$
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Game OW :
$(sk, pk) \leftarrow \mathcal{K}();$
$y \xleftarrow{\$} \{0,1\}^\ell;$
$y' \leftarrow \mathcal{I}(f_{pk}(y));$
return $y = y'$
Adversary $\mathcal{I}(x)$:
$(m_0, m_1) \leftarrow \mathcal{A}_1(pk);$
$s \xleftarrow{\$} \{0,1\}^k;$
$c^\star \leftarrow x \,\|\, s;$
$b' \leftarrow \mathcal{A}_2(c^\star);$
$y' \leftarrow [z \in L_H^{\mathcal{A}} \mid f_{pk}(z) = x];$
return y'

1. For each hop
   - prove validity of pRHL judgment
   - derive probability claims
   - (possibly) resolve some probability expressions using pHL
2. Obtain security bound by combining claims
3. Check execution time of constructed adversary

Conditional equivalence

$\mathcal{E}_{pk}(m)$:
$r \xleftarrow{\$} \{0,1\}^{\ell}$;
$h \leftarrow H(r)$;
$s \leftarrow h \oplus m$;
$c \leftarrow f_{pk}(r) \,\|\, s$;
return c

$\mathcal{E}_{pk}(m)$:
$r \xleftarrow{\$} \{0,1\}^{\ell}$;
$h \xleftarrow{\$} \{0,1\}^{k}$;
$s \leftarrow h \oplus m$;
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$$\models \{\top\} \text{ IND-CPA } \sim \mathbf{G} \left\{ (\neg r \in L_H^{\mathcal{A}})\langle 2 \rangle \rightarrow =_{b,b'} \right\}$$

$$\Pr_{\text{IND-CPA}}\left[b' = b\right] - \Pr_{\mathbf{G}}\left[b' = b\right] \leq \Pr_{\mathbf{G}}\left[r \in L_H^{\mathcal{A}}\right]$$

Equivalence

$\mathcal{E}_{pk}(m)$:
$r \xleftarrow{\$} \{0,1\}^{\ell}$;
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$s \leftarrow h \oplus m$;
$c \leftarrow f_{pk}(r) \| s$;
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$$\models \{\top\}\ \mathbf{G}\ \sim\ \mathbf{G}'\ \left\{=_{b,b',r,L_H^{\mathcal{A}}}\right\}$$

$$\Pr_{\mathbf{G}}\left[r \in L_H^{\mathcal{A}}\right] = \Pr_{\mathbf{G}'}\left[r \in L_H^{\mathcal{A}}\right] \qquad \Pr_{\mathbf{G}}[b' = b] = \Pr_{\mathbf{G}'}[b' = b] = \frac{1}{2}$$

Equivalence

$\mathcal{E}_{pk}(m)$:
$r \xleftarrow{\$} \{0,1\}^{\ell}$;
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$$\models \{\top\}\ \mathbf{G}\ \sim\ \mathbf{G}'\ \left\{=_{b,b',r,L_H^{\mathcal{A}}}\right\}$$

$$\Pr_{\text{IND-CPA}}[b' = b] - \tfrac{1}{2} \leq \Pr_{\mathbf{G}'}\left[r \in L_H^{\mathcal{A}}\right]$$

Reduction

Game INDCPA :
$(sk, pk) \leftarrow \mathcal{K}();$
$(m_0, m_1) \leftarrow \mathcal{A}_1(pk);$
$b \xleftarrow{\$} \{0,1\};$
$c^\star \leftarrow \mathcal{E}_{pk}(m_b);$
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return $(b' = b)$
Encryption $\mathcal{E}_{pk}(m)$:
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return c

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$(sk, pk) \leftarrow \mathcal{K}();$
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$y' \leftarrow \mathcal{I}(f_{pk}(y));$
return $y = y'$
Adversary $\mathcal{I}(x)$:
$(m_0, m_1) \leftarrow \mathcal{A}_1(pk);$
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$s \xleftarrow{\$} \{0,1\}^k;$
$c^\star \leftarrow x \,\|\, s;$
$b' \leftarrow \mathcal{A}_2(c^\star);$
$y' \leftarrow [z \in L_H^{\mathcal{A}} \mid f_{pk}(z) = x];$
return y'

$$\models \{\top\}\ \mathbf{G}'\ \sim\ \mathrm{OW}\ \left\{(r \in L_H^{\mathcal{A}})\langle 1\rangle \rightarrow (y' = y)\langle 2\rangle\right\}$$

$$\Pr_{\mathbf{G}'}\left[r \in L_H^{\mathcal{A}}\right] \leq \Pr_{\mathrm{OW}(\mathcal{I})}[y' = y]$$

Reduction

Game INDCPA :
$(sk, pk) \leftarrow \mathcal{K}();$
$(m_0, m_1) \leftarrow \mathcal{A}_1(pk);$
$b \xleftarrow{\$} \{0,1\};$
$c^\star \leftarrow \mathcal{E}_{pk}(m_b);$
$b' \leftarrow \mathcal{A}_2(c^\star);$
return $(b' = b)$

Encryption $\mathcal{E}_{pk}(m)$:
$r \xleftarrow{\$} \{0,1\}^\ell;$
$s \xleftarrow{\$} \{0,1\}^k;$
$c \leftarrow f_{pk}(r) \,\|\, s;$
return c

Game OW :
$(sk, pk) \leftarrow \mathcal{K}();$
$y \xleftarrow{\$} \{0,1\}^\ell;$
$y' \leftarrow \mathcal{I}(f_{pk}(y));$
return $y = y'$

Adversary $\mathcal{I}(x)$:
$(m_0, m_1) \leftarrow \mathcal{A}_1(pk);$
$b \xleftarrow{\$} \{0,1\};$
$s \xleftarrow{\$} \{0,1\}^k;$
$c^\star \leftarrow x \,\|\, s;$
$b' \leftarrow \mathcal{A}_2(c^\star);$
$y' \leftarrow [z \in L_H^{\mathcal{A}} \mid f_{pk}(z) = x];$
return y'

$$\models \{\top\}\ \mathbf{G}'\ \sim\ \mathrm{OW}\ \left\{(r \in L_H^{\mathcal{A}})\langle 1\rangle \rightarrow (y' = y)\langle 2\rangle\right\}$$

$$\Pr_{\text{IND-CPA}(\mathcal{A})}[b' = b] - \tfrac{1}{2} \leq \Pr_{\text{OW}(\mathcal{I})}[y' = y]$$

EasyCrypt

Domain-specific proof assistant

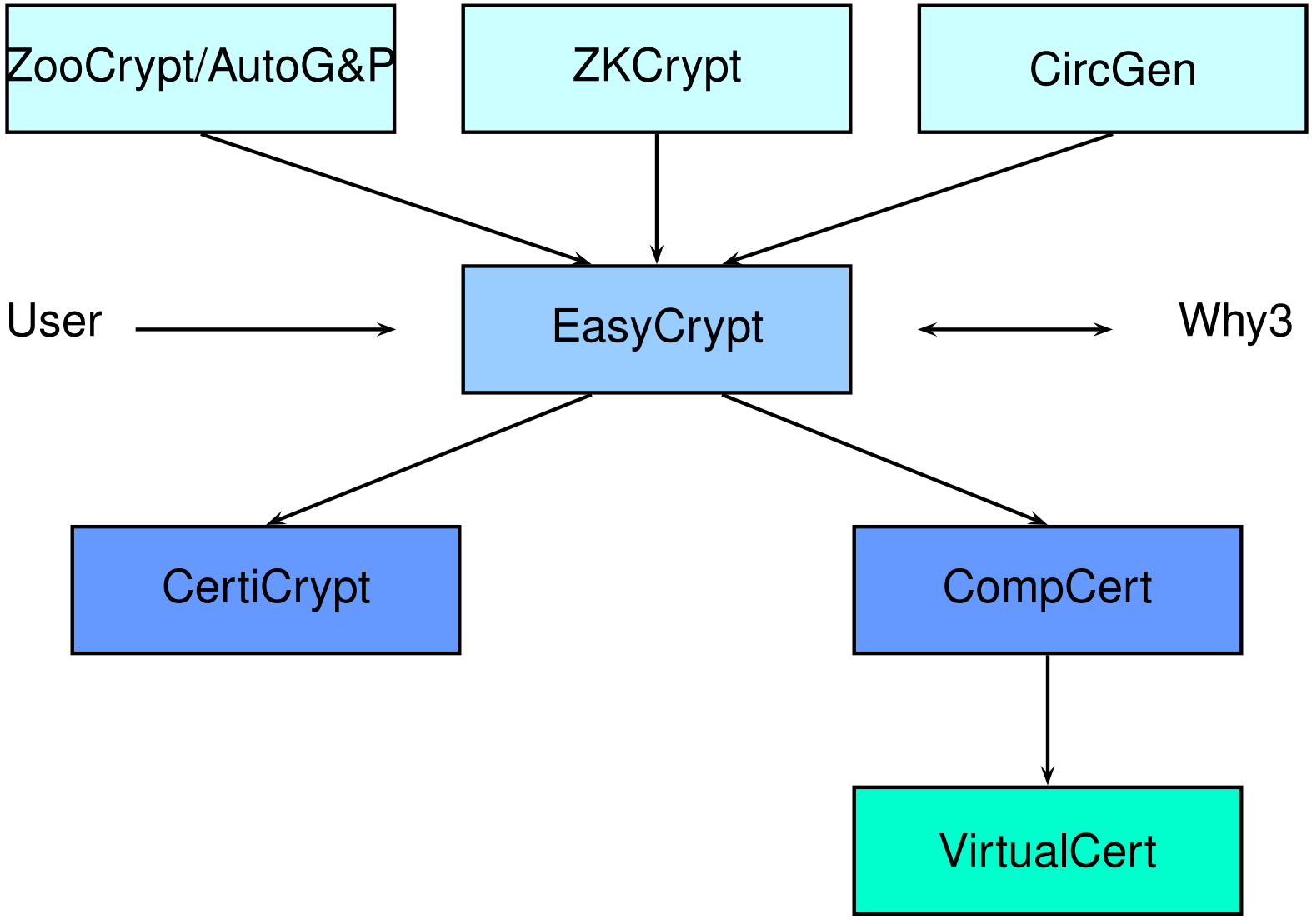
- proof goals tailored to reductionist proofs
- proof tools support common proof techniques (bridging steps, failure events, hybrid arguments, eager sampling. . .)

Control and automation from state-of-art verification

- interactive proof engine and mathematical libraries (a la Coq/ssreflect)
- back-end to SMT solvers and CAS

Many case studies:

- Encryption, signatures, hash designs, key exchange protocols, zero-knowledge protocols, garbled circuits, SHA3, voting


ZooCrypt/AutoG&P
ZKCrypt
CircGen
User
EasyCrypt
Why3
CertiCrypt
CompCert
VirtualCert

Summary

- Solid foundation for cryptographic proofs
- Formal verification of emblematic case studies
- Further work: automation

`http://www.easycrypt.info`