Symmetry and Asymmetry in the Logical Laws

Rereading Dummett through Girard [and vice versa!]

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Part I

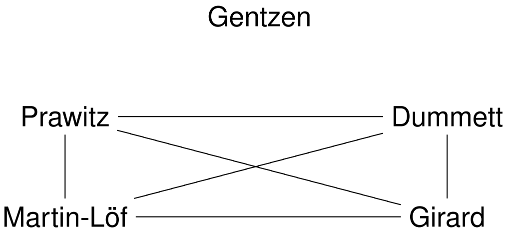
Reading Dummett

In the beginning. . .

The introductions represent, as it were, 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. . .

[Gentzen, Untersuchungen 1, §5.13]

Exegesis



Two of Dummett's (Many) Insights

One insight, a refinement of Prawitz:

A more restrictive notion of canonical proof
justifies more general inferences

One insight, a heresy against Gentzen:

The elimination rules can also be seen as fixing
the meanings of the connectives

(These two insights are in fact strongly related.)

The First Insight

A more restrictive notion of canonical proof justifies more general inferences

$$\dfrac{\dfrac{\overset{\vdots}{A} \qquad \dfrac{\overset{\vdots}{C}}{B \vee C}}{A \wedge (B \vee C)} \qquad \begin{matrix}[A][B]\\ D\end{matrix} \qquad \begin{matrix}[A][C]\\ D\end{matrix}}{D} \qquad \leadsto \qquad \begin{matrix} \overset{\vdots}{A} \quad \overset{\vdots}{C} \\ \vdots \\ D \end{matrix}$$

The Second Insight

The elimination rules can also be seen as fixing the meanings of the connectives

$$\begin{array}{c}\dfrac{\dfrac{A \quad B}{A \wedge B}}{A} \\ \vdots\end{array} \;\leadsto\; \begin{array}{c} A \\ \vdots\end{array} \qquad\qquad \begin{array}{c}\dfrac{\dfrac{A \quad B}{A \wedge B}}{B} \\ \vdots\end{array} \;\leadsto\; \begin{array}{c} B \\ \vdots\end{array}$$

Historical Aside

Intuitively, Gentzen's suggestion that the introduction rules be viewed as fixing the meanings of the logical constants has no more force than the converse suggestion, that they are fixed by the elimination rules; intuitive plausibility oscillates between these opposing suggestions as we move from one logical constant to another. **Per Martin-Löf has, indeed, constructed an entire meaning-theory for the language of mathematics on the basis of the assumption that it is the elimination rules that determine meaning.**

[TLBM, p. 280, emphasis mine]

Historical Aside, cont.

To explain the meaning of an implication $A \supset B$, we must explain what is the purpose (function, role) of a canonical proof of $A \supset B$. And, specializing the explanation given above [for the dependent product $(\Pi u \in A)B(u)$], this purpose is to be applied to a canonical proof of the proposition denoted by A, thereby yielding a canonical proof of the proposition denoted by B. **In no way is it correct to say that the meaning of $A \supset B$ is determined by the introduction rule.**

[Letter from M-L to D, 5 March 1976, emphasis mine]

(compare, e.g., the 1983 Siena Lectures)

Problem 1

The verificationist meaning-theory prima facie justifies. . .

$$\frac{\forall x.A(x) \vee B(x)}{[\forall x.A(x)] \vee [\forall x.B(x)]} \qquad \frac{A \supset (B \vee C)}{(A \supset B) \vee (A \supset C)}$$

Solution 1

. . . so just patch the notion of canonical proof for $\forall$ and $\supset$

> *The proof-theoretic justification procedure itself is elegant; but, in vindicating its applicability to arguments within empirical discourse, we have had to exchange this elegance for an unattractive messiness.* [279]

Problem 2

Hard to decide between verificationist and pragmatist...

> *It is easier to acknowledge this general distinction between two aspects of the use of sentences than to apply it to specific expressions.* [212]

Solution 2

. . . so leave it for another day. . .

> *Our immediate concern is not with the question which, if either, of these two aspects of our use of sentences should be taken as the central notion of the meaning-theory, but with the mere fact that linguistic practice has these two aspects.* [214]

> *The laws of intuitionistic logic appear capable of being justified proof-theoretically by any of the procedures we have discussed.* [299]

Another Day

Dummett's analysis in light of Linear Logic

(but without the linearity)

Part II

Classical Polarity

Focusing Proofs for Linear Logic

Andreoli's miraculous proof-search strategy

$\otimes, \oplus, \exists$	vs.	$⅋, \&, \forall$
focus		inversion

A New Constructive Logic: Classical Logic

Girard's insight: can explicitly **polarize** the classical connectives

...decomposing the classical $\neg\neg$ translations...

...and hence explaining the different ways in which classical logic can be endowed with constructive content.

Polarity in light of Dummett

Claim: Polarity is *precisely* the V/P distinction

Polarity in light of Dummett

(Actually, the claim is a bit of a lie.)

Classical polarity is dumbed-down Dummett

(Let's illustrate. . .)

Judgments

Basic judgments:

Judgment J	Natural reading	Sequent reading
$[P]$	"P obvious"	right-focus
$\bullet P$	"P false"	left-inversion
N	"N true"	right-inversion
$\bullet[N]$	"N absurd"	left-focus
$\#$	contradiction	neutral
Δ	conjunction of Δ	multiple

where $\Gamma, \Delta ::= \cdot \mid (\Gamma, \Delta) \mid \bullet P \mid N$

Dictionaries

A **dictionary** is a pair of relations...

$$\Delta \Vdash [P] \qquad \Delta \Vdash \bullet[N]$$

...*defining* some positive and negative connectives

Derivations of $\Delta \Vdash [P]$ and $\Delta \Vdash \bullet[N]$ are called **patterns**

We do not care about the contents of the dictionary

More D. . .

Someone who has not opted for any particular theory of meaning, whether verificationist or pragmatist, but wants to characterise our understanding of the logical constants in terms of our mastery of the use of sentences containing them, is likely to invoke the introduction rule for the existential quantifier and the elimination rule for the the universal one. Wittgenstein, for instance, does precisely this in scattered places in his writing. But this can hardly be meant as more than illustrative. No one can be said to understand either quantifier unless he at least knows both the introduction and elimination rules for it: **only a systematic theory, which will provide for the derivation of all other features of use from that which has been selected as the central notion of the theory**, *can afford to pick out one or the other type of rule as the distinguished determinant of meaning.* [217, emphasis mine]

Proof and Refutation

$$\frac{\Delta \Vdash [P] \quad \Gamma \vdash \Delta}{\Gamma \vdash [P]} \qquad \frac{\Delta \Vdash [P] \quad \longrightarrow \quad \Gamma, \Delta \vdash \#}{\Gamma \vdash \bullet P}$$

$$\frac{\Delta \Vdash \bullet[N] \quad \longrightarrow \quad \Gamma, \Delta \vdash \#}{\Gamma \vdash N} \qquad \frac{\Delta \Vdash \bullet[N] \quad \Gamma \vdash \Delta}{\Gamma \vdash \bullet[N]}$$

Conjunction and Contradiction

$$\frac{}{\Gamma \vdash \cdot} \qquad \frac{\Gamma \vdash \Delta_1 \quad \Gamma \vdash \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2}$$

$$\frac{\bullet P \in \Gamma \quad \Gamma \vdash [P]}{\Gamma \vdash \#} \qquad \frac{N \in \Gamma \quad \Gamma \vdash \bullet[N]}{\Gamma \vdash \#}$$

Constructions on Derivations

Principle (Identity)

We can always witness[*]...

1. $\Gamma(\Delta) \vdash \Delta$

Principle (Composition)

We can always witness[†]...

1. *If* $\Gamma \vdash [P]$ *and* $\Gamma \vdash \bullet P$ *then* $\Gamma \vdash \#$
2. *If* $\Gamma \vdash N$ *and* $\Gamma \vdash \bullet[N]$ *then* $\Gamma \vdash \#$
3. *If* $\Gamma \vdash \Delta$ *and* $\Gamma(\Delta) \vdash J$ *then* $\Gamma \vdash J$

[$*$/†]: if the dictionary contains recursive definitions, we may need to give non-wellfounded/partial derivations

Justification Procedures

The following rules can be *justified*. . .

$$\frac{\Gamma \vdash [P_1] \quad \Gamma \vdash [P_2]}{\Gamma \vdash [P_1 \otimes P_2]} \qquad \frac{\Gamma \vdash N_1 \quad \Gamma \vdash N_2}{\Gamma \vdash N_1 \& N_2}$$

. . . in quite different ways

Intuitionistic interpretation

Translate (e.g., a few standard) polarized formulas P^+ and N^-:

$$(P_1 \otimes P_2)^+ = P_1^+ \wedge P_2^+ \qquad (P_1 \oplus P_2)^+ = P_1^+ \vee P_2^+$$

$$(N_1 \& N_2)^- = N_1^- \vee N_2^- \qquad (N_1 \invamp N_2)^- = N_1^- \wedge N_2^-$$

$$(P \to N)^- = P^+ \wedge N^- \qquad (N^\perp)^+ = N^-$$

$$(\uparrow P)^- = \neg P^+ \qquad (\downarrow N)^+ = \neg N^-$$

where $\neg A = A \supset \#$, for some distinguished atom $\#$

Intuitionistic interpretation

Translate judgments J^*:

$$[P]^* = P^+ \quad N^* = \neg N^- \quad \bullet[N]^* = N^- \quad \bullet P^* = \neg P^+$$

$$\cdot^* = \mathrm{T} \quad (\Delta_1, \Delta_2)^* = \Delta_1^* \wedge \Delta_2^* \quad \#^* = \#$$

Theorem

$\Gamma \vdash J$ *iff* $\Gamma^* \vdash^i J^*$

Classical interpretation (polarization)

Define “polarity-collapsing” translation:

$$|\otimes| = |\&| = \wedge \quad |\oplus| = |\oplus| = \vee \quad |\rightarrow| = \supset \quad |-^{\perp}| = \neg \quad |\downarrow| = |\uparrow| = \cdot$$

Theorem

$\Gamma \vdash \#$ *iff* $|\Gamma| \vdash^{c} \#$

Proof: compose $A^* \equiv^c |A|$ with Glivenko’s theorem

Punchline: recover different ¬¬-interpretations of classical logic

Part III

Generalized Polarity

Over-Symmplification

Our analysis was a **caricature** of Dummett's. . .

. . . can we take *seriously* the idea of a pragmatist?

(As opposed to a falsificationist)

A Positive Answer

Idea: generalize “contradiction” $\#$ to arbitrary consequence P

...and refutation $\bullet(-)$ to “generalized refutation” $(-) \triangleright P$

And thus break the perfect symmetry between positive and negative!

(related by C-H to “delimited continuations”)

Recall: Proof and Refutation

$$\frac{\Delta \Vdash [P] \quad \Gamma \vdash \Delta}{\Gamma \vdash [P]} \qquad \frac{\Delta \Vdash [P] \quad \longrightarrow \quad \Gamma, \Delta \vdash \#}{\Gamma \vdash \bullet P}$$

$$\frac{\Delta \Vdash \bullet[N] \quad \longrightarrow \quad \Gamma, \Delta \vdash \#}{\Gamma \vdash N} \qquad \frac{\Delta \Vdash \bullet[N] \quad \Gamma \vdash \Delta}{\Gamma \vdash \bullet[N]}$$

Generalized* Proof and Refutation

$$\frac{\Delta \Vdash [P] \quad \Gamma \vdash \Delta}{\Gamma \vdash [P]} \qquad \frac{\Delta \Vdash [P] \quad \longrightarrow \quad \Gamma, \Delta \vdash P_r}{\Gamma \vdash P \triangleright P_r}$$

$$\frac{\Delta \Vdash [N] \triangleright P_r \quad \longrightarrow \quad \Gamma, \Delta \vdash P_r}{\Gamma \vdash N} \qquad \frac{\Delta \Vdash [N] \triangleright P_r \quad \Gamma \vdash \Delta}{\Gamma \vdash [N] \triangleright P_r}$$

Generalized Proof and Refutation

$$\frac{\Delta \Vdash [P] \quad \Gamma \vdash \Delta}{\Gamma \vdash [P]} \qquad \frac{\Delta \Vdash [P] \quad \longrightarrow \quad \Gamma, \Delta \vdash P_r}{\Gamma \vdash P \triangleright P_r}$$

$$\frac{\alpha.\Delta \Vdash [N] \triangleright - \quad \longrightarrow \quad \Gamma, \Delta \vdash \alpha}{\Gamma \vdash N} \qquad \frac{\alpha.\Delta \Vdash [N] \triangleright - \quad \Gamma \vdash \Delta[P_r/\alpha]}{\Gamma \vdash [N] \triangleright P_r}$$

Recall: Contradiction

$$\frac{\bullet P \in \Gamma \quad \Gamma \vdash [P]}{\Gamma \vdash \#} \qquad \frac{N \in \Gamma \quad \Gamma \vdash \bullet[N]}{\Gamma \vdash \#}$$

Generalized** Contradiction

$$\frac{P \triangleright P_r \in \Gamma \quad \Gamma \vdash [P]}{\Gamma \vdash P_r} \qquad \frac{N \in \Gamma \quad \Gamma \vdash [N] \triangleright P_r}{\Gamma \vdash P_r}$$

Generalized* Contradiction

$$\frac{P \triangleright P_{r'} \in \Gamma \quad \Gamma \vdash [P] \quad \Gamma \vdash P_{r'} \triangleright P_r}{\Gamma \vdash P_r} \qquad \frac{N \in \Gamma \quad \Gamma \vdash [N] \triangleright P_r}{\Gamma \vdash P_r}$$

Generalized Contradiction

$$\frac{\Gamma \vdash [P]}{\Gamma \vdash P} \qquad \frac{P \triangleright P_{r'} \in \Gamma \quad \Gamma \vdash [P] \quad \Gamma \vdash P_{r'} \triangleright P_r}{\Gamma \vdash P_r} \qquad \frac{N \in \Gamma \quad \Gamma \vdash [N] \triangleright P_r}{\Gamma \vdash P_r}$$

Constructions on Derivations

Principle (Identity)

We can always witness*...

1. $\Gamma(\Delta) \vdash \Delta$
2. $\Gamma \vdash P \triangleright P$

Principle (Composition)

We can always witness†...

1. If $\Gamma \vdash [P]$ and $\Gamma \vdash P \triangleright P_r$ then $\Gamma \vdash P_r$
2. If $\Gamma \vdash N$ and $\Gamma \vdash [N] \triangleright P_r$ then $\Gamma \vdash P_r$
3. If $\Gamma \vdash P$ and $\Gamma \vdash P \triangleright P_r$ then $\Gamma \vdash P_r$
4. If $\Gamma \vdash \Delta$ and $\Gamma(\Delta) \vdash J$ then $\Gamma \vdash J$
5. If $\Gamma \vdash \Delta[P_r/\alpha]$ and $\Gamma(\Delta) \vdash J$ then $\Gamma \vdash J[P_r/\alpha]$
6. If $\Gamma \vdash P \triangleright P_{r'}$ and $\Gamma \vdash P_{r'} \triangleright P_r$ then $\Gamma \vdash P \triangleright P_r$
7. If $\Gamma \vdash [N] \triangleright P_{r'}$ and $\Gamma \vdash P_{r'} \triangleright P_r$ then $\Gamma \vdash [N] \triangleright P_r$

Second-order intuitionistic interpretation

Translate (a few standard++) polarized formulas P^+ and $N^{-\alpha}$:

$$(P_1 \otimes P_2)^+ = P_1^+ \wedge P_2^+ \qquad (P_1 \oplus P_2)^+ = P_1^+ \vee P_2^+$$

$$(N_1 \& N_2)^{-\alpha} = N_1^{-\alpha} \vee N_2^{-\alpha} \qquad (N_1 \parr N_2)^{-\alpha} = N_1^{-\alpha} \wedge N_2^{-\alpha}$$

$$(P \to N)^{-\alpha} = P^+ \wedge N^{-\alpha} \qquad (N \multimapdot P)^+ = N^{-\alpha}[P^+/\alpha]$$

$$(\uparrow P)^{-\alpha} = P^+ \supset \alpha \qquad (\downarrow N)^+ = \forall \alpha . N^{-\alpha} \supset \alpha$$

Second-order intuitionistic interpretation

Translate judgments J^*:

$$[P]^* = P^+ \qquad N^* = \forall\alpha.N^{\text{-}\alpha} \supset \alpha$$

$$[N] \triangleright P_r{}^* = N^{\text{-}\alpha}[P_r^*/\alpha] \qquad P \triangleright P_r{}^* = P^+ \supset P_r^*$$

$$\cdot^* = \mathrm{T} \quad (\Delta_1, \Delta_2)^* = \Delta_1^* \wedge \Delta_2^* \quad P^* = P^+$$

Theorem

$\Gamma \vdash A$ *iff* $\Gamma^* \vdash^{\forall 2i} A^*$

Intuitionistic interpretation (polarization)

Again define "polarity-collapsing" translation:

$$|\otimes| = |\&| = \wedge \quad |\oplus| = |\oplus| = \vee \quad |\to| = |{-\!\bullet}| = \supset \quad |\downarrow| = |\uparrow| = \cdot$$

Terminology: **pure** = $-\!\bullet$-free, **orderly** = $\otimes$-free, **immaculate** = both

Theorem

For immaculate Γ and A, $\Gamma \vdash A$ iff $|\Gamma| \vdash^i |A|$

Proof: $A^* \equiv^{\forall 2i} |A|$ (quantifier commutations + "Friedman's trick")

Important point: ¬immaculate $\not\Rightarrow$ **immoral**

Conclusions and Questions

Dummett-style PTS is intimately related to “French” proof-theory

Polarity is an important basis for finding meaning in logic

Asymmetry is still not very well-understood

Is generalized polarity observable in natural language semantics?

References

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