# Static Profiling of Parametric Resource Usage as a Valuable Aid for Hot-spot Detection

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# Introduction and Motivation

- Resources: non-func. numerical properties about the execution of a program.
  - Examples: resolution steps, execution time, energy consumption, # of calls to a predicate, # of network accesses, # of transactions, ...
- Goal of static analysis:
  estimating the resource usage of the execution of a program without running it with concrete data, as function of input data sizes and possibly other parameters.

  Typical size metrics → actual value of a number, the length of a list, the number of constant and function symbols of a term, etc.
- Significant work done in logic programming.
  - + Allows analysis of other languages via transformation into Horn Clauses.
- Resource analysis is very useful:
  - Automatic program optimization.
  - Verification of resource-related specifications.
  - Detection of performance bugs, help guiding software design, ...
    Example: developing energy-efficient software.

# Inferring Accumulated Cost [TPLP'16, FLOPS'16]

- Helping developers make (resource-related) design decisions:
  - Which parts of the program are the most resource-consuming?
  - Which predicates should be optimized first?
- The standard/classical notion of cost only partially meets these objectives:
  - Predicates w/highest (standard) costs may not need to be optimized first.
  - E.g., perhaps predicates with lower costs but which are called more often.
  - The input sizes to such calls are also relevant.
- Need info resulting from a static profiling of the program to:
  - identify the parts of a program responsible for highest fractions of the cost $\rightarrow$ accumulated cost.
  - I.e., how the total resource usage of the execution of a program is *distributed* over selected parts of it (cost centers $\rightarrow$ predicates).

  Static profiling $\rightarrow$ *static inference of the kinds of information that are usually obtained at run-time by profilers*.

## Main contribution

Novel, general, and flexible framework for setting up cost equations/relations.
$\rightarrow$ can be instantiated for performing a wide range of static resource usage analyses, including both accumulated cost and standard cost.

# Overview of the Classical Cost Analysis

1. Perform all the required supporting analyses (examples):
   - Types (shapes) for inferring size metrics (list-length, term-depth, ...).
   - Mode analysis to determine input/output arguments.
   - Sharing analysis for correctness (conservative: only when there is no sharing among data structures).
   - *Non-failure* (no exceptions) inferred for non-trivial lower bounds.
   - *Determinacy* (mutual exclusion) to obtain tighter bounds.
2. Set up recurrence equations representing the size of each (relevant) output argument as a function of the input data sizes.
   - Size metrics are derived from inferred type (shape) information.
   - Data dependency graphs used to determine *relative* sizes of variable contents.
3. Compute (lower/upper) bounds to the solutions of these recurrence equations to obtain output argument sizes as (closed-form) functions of input sizes.
   - Using internal recurrence solver, or the interfaces with Mathematica, Parma, PUBS, Matlab, etc.

[PLDI'90, SAS'94, PASCO'94]

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4. E.g.:

```
:- true pred append(A,B,C) : list * list * var
                       => ( size_lb(C, length(A)+length(B)),
                            size_ub(C, length(A)+length(B)) ).
```

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5. Use the size information to set up recurrence equations representing the computational cost of each clause and compute bounds to their solutions to obtain **cost functions**.

[PLDI'90, SAS'94, PASCO'94]

# Size Metrics

- Various size metrics can be used to determine the size of an input:
  - the actual value of a number,
  - the length of a list,
  - the number of constant and function symbols in a term.
  - the depth of a term,
  - etc.
- These are automatically inferred based on type (shape) analysis and other information (program control flow and operations).
- The function $size_m(t)$ defines the size of a term $t$ under the metric $m$:
  $size_{length}(\texttt{[4,2,7]}) = 3$
  $size_{length}(\texttt{[]}) = 0$
  $size_{term\_depth}(\texttt{f(a,g(b))}) = 2$
- The function $diff_m(t_1, t_2)$ gives the size difference between two terms $t_1$ and $t_2$ under the metric $m$:
  $diff_{length}([2, 3|\texttt{L}], [4|\texttt{L}]) = 1$
  $diff_{length}(\texttt{L}, [\texttt{H}|\texttt{L}]) = -1$
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# Size Analysis (size relations): Example

```
:- entry nrev/2 : list(num) * var.

nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).

app([],L,L).
app([H|L],L1,[H|R]) :- app(L,L1,R).
```

- The automatically inferred size metric is *length* (list length) for all arguments.
- All arguments inferred to be input except last ones (output). No aliasing.
- Let $\langle b.i \rangle$ denote (a bound on) the size of the term(s) appearing in the $i^{th}$ argument position in the head of a clause defining predicate $b$.
- Let $\langle p.j.i \rangle$ denote (a bound on) the size of the term(s) appearing in the $i^{th}$ argument position in the $j^{th}$ body call of a clause defining predicate $b$.

  $\rightarrow$ $p$ denotes the predicate called (for readability).
- Example (2nd clause): $\langle app.1.1 \rangle = length($`L`$)$ and $\langle app.1 \rangle = length($`[H|L]`$)$.
- First, we consider predicate `app(A,B,C)` (third arg is output).
- We want to obtain intra-predicate argument size relations: $Sz_3^{app}(x, y)$ represents the size of the third argument of `app` as a function of its input data sizes ($x = length($`A`$)$ and $y = length($`B`$)$).

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  $Sz_3^{app}(n, m) = m \quad \text{if } n = 0$
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  are solved, obtaining the closed-form function:
  $Sz_3^{app}(n, m) = n + m \quad \text{if } n \geq 0$

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  $Sz_3^{app}(n, m) = Sz_3^{app}(n-1, m) + 1 \quad \text{if } n > 0$
  are solved, obtaining the closed-form function:
  $Sz_3^{app}(n, m) = n + m \quad \text{if } n \geq 0$
  which is used for the analysis of predicate `nrev`.

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:
  $\langle \mathit{nrev}.1.1 \rangle = \langle \mathit{nrev}.1 \rangle + \mathit{diff}(\mathtt{L}, \mathtt{[H|L]})$ *(inter-predicate)*

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:
  $\mathit{length}(\texttt{L}) = \mathit{length}(\texttt{[H|L]}) + \mathit{diff}(\texttt{L}, \texttt{[H|L]})$

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:
  $\mathit{length}(\texttt{L}) = \mathit{length}(\texttt{[H|L]}) - 1$

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:
  $\langle nrev.1.1\rangle = \langle nrev.1\rangle - 1 \quad \equiv \quad length(\mathtt{L}) = length(\mathtt{[H|L]}) - 1$

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:
  $\langle nrev.1.1\rangle = \langle nrev.1\rangle - 1$

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
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- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:
  $\langle nrev.1.1 \rangle = \langle nrev.1 \rangle - 1$
  $\langle nrev.1.2 \rangle = Sz_2^{nrev}(\langle nrev.1.1 \rangle)$ *(intra-predicate)*

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:
  $\langle nrev.1.1\rangle = \langle nrev.1\rangle - 1$
  $\langle nrev.1.2\rangle = Sz_2^{nrev}(\langle nrev.1.1\rangle) \quad \equiv \quad length(\mathtt{R1}) = Sz_2^{nrev}(length(\mathtt{L}))$

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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  $\langle nrev.1.1 \rangle = \langle nrev.1 \rangle - 1$
  $\langle nrev.1.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$ *(normalizing)*

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:
  $\langle nrev.1.1 \rangle = \langle nrev.1 \rangle - 1$
  $\langle nrev.1.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$
  $\langle app.2.1 \rangle = \langle nrev.1.2 \rangle$ *(inter-predicate)*

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:
  $\langle nrev.1.1 \rangle = \langle nrev.1 \rangle - 1$
  $\langle nrev.1.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$
  $\langle app.2.1 \rangle = \langle nrev.1.2 \rangle \quad \equiv \quad length(\texttt{R1}) = length(\texttt{R1})$

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
- Argument size relations for the recursive clause:

  $\langle nrev.1.1 \rangle = \langle nrev.1 \rangle - 1$

  $\langle nrev.1.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$

  $\langle app.2.1 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$ *(normalizing)*

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
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- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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  $\langle nrev.1.1 \rangle = \langle nrev.1 \rangle - 1$
  $\langle nrev.1.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$
  $\langle app.2.1 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$
  $\langle app.2.2 \rangle = length($`[H]`$)$ *(explicit size)*

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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  $\langle nrev.1.1 \rangle = \langle nrev.1 \rangle - 1$

  $\langle nrev.1.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$

  $\langle app.2.1 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$

  $\langle app.2.2 \rangle = 1$

# Size Analysis (size relations): Example

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nrev([],[]).
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- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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$$\langle nrev.1.1 \rangle = \langle nrev.1 \rangle - 1$$
$$\langle nrev.1.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$$
$$\langle app.2.1 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$$
$$\langle app.2.2 \rangle = 1$$
$$\langle app.2.3 \rangle = Sz_3^{app}(\langle app.2.1 \rangle, \langle app.2.2 \rangle) \quad \textit{(intra-predicate)}$$

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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$$\langle nrev.1.1\rangle = \langle nrev.1\rangle - 1$$
$$\langle nrev.1.2\rangle = Sz_2^{nrev}(\langle nrev.1\rangle - 1)$$
$$\langle app.2.1\rangle = Sz_2^{nrev}(\langle nrev.1\rangle - 1)$$
$$\langle app.2.2\rangle = 1$$
$$\langle app.2.3\rangle = \langle app.2.1\rangle + \langle app.2.2\rangle \quad \textit{using} \quad Sz_3^{app}(x, y) = x + y$$

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
```

- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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$$\langle nrev.1.1\rangle = \langle nrev.1\rangle - 1$$
$$\langle nrev.1.2\rangle = Sz_2^{nrev}(\langle nrev.1\rangle - 1)$$
$$\langle app.2.1\rangle = Sz_2^{nrev}(\langle nrev.1\rangle - 1)$$
$$\langle app.2.2\rangle = 1$$
$$\langle app.2.3\rangle = Sz_2^{nrev}(\langle nrev.1\rangle - 1) + 1 \quad \textit{(normalizing)}$$

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
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- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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  $\langle app.2.1\rangle = Sz_2^{nrev}(\langle nrev.1\rangle - 1)$
  $\langle app.2.2\rangle = 1$
  $\langle app.2.3\rangle = Sz_2^{nrev}(\langle nrev.1\rangle - 1) + 1$
  $\langle nrev.2\rangle = \langle app.2.3\rangle + diff(\mathtt{R}, \mathtt{R})$ *(inter-predicate)*

# Size Analysis (size relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
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$$\langle app.2.2 \rangle = 1$$
$$\langle app.2.3 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$$
$$\langle nrev.2 \rangle = \langle app.2.3 \rangle + 0$$

# Size Analysis (size relations): Example

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nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
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- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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  $Sz_2^{nrev}(\langle nrev.1 \rangle) = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$
- From the first clause of `nrev`, we obtain the equation:
  $Sz_2^{nrev}(0) = 0$

# Size Analysis (size relations): Example

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nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
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- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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  $\langle app.2.1 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$
  $\langle app.2.2 \rangle = 1$
  $\langle app.2.3 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$
  $\langle nrev.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$
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  $Sz_2^{nrev}(0) = 0$
- The equations:

  $Sz_2^{nrev}(0) = 0$
  $Sz_2^{nrev}(\langle nrev.1 \rangle) = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$

# Size Analysis (size relations): Example

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nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
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- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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  $\langle app.2.1 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$
  $\langle app.2.2 \rangle = 1$
  $\langle app.2.3 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$
  $\langle nrev.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$
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- From the first clause of `nrev`, we obtain the equation:
  $Sz_2^{nrev}(0) = 0$
- The equations ($n = \langle nrev.1 \rangle$):
  $Sz_2^{nrev}(0) = 0$
  $Sz_2^{nrev}(n) = Sz_2^{nrev}(n-1) + 1$

# Size Analysis (size relations): Example

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nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).
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- We now switch to predicate `nrev(A, B)`, where `A` and `B` are input and output arguments respectively.
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  $\langle nrev.1.1 \rangle = \langle nrev.1 \rangle - 1$
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  $\langle app.2.1 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1)$
  $\langle app.2.2 \rangle = 1$
  $\langle app.2.3 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$
  $\langle nrev.2 \rangle = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$
  $Sz_2^{nrev}(\langle nrev.1 \rangle) = Sz_2^{nrev}(\langle nrev.1 \rangle - 1) + 1$
- From the first clause of `nrev`, we obtain the equation:
  $Sz_2^{nrev}(0) = 0$
- The equations ($n = \langle nrev.1 \rangle$):
  $Sz_2^{nrev}(0) = 0$
  $Sz_2^{nrev}(n) = Sz_2^{nrev}(n-1) + 1$
  are solved, obtaining the closed-form function:
  $Sz_2^{nrev}(n) = n$

# Size Analysis (size relations): Example

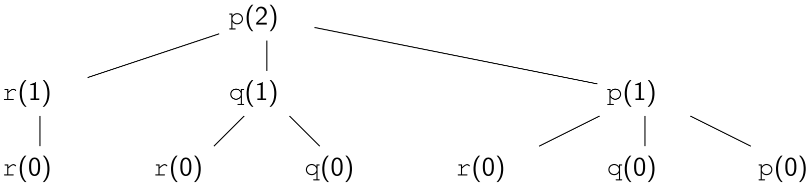
- The size of the output argument of `nrev(A, B)` is given by the following equations (where $n = length(\mathtt{A})$):
  $Sz_2^{nrev}(0) = 0$
  $Sz_2^{nrev}(n) = Sz_2^{nrev}(n-1) + 1$
- Solution: $Sz_2^{nrev}(n) = n$.
  The length (size) of the output argument of `nrev` is equal to the length of its input.

# Standard Cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

The standard cost of a call $\mathtt{p}(2)$ (in number of resolution steps): $\mathtt{C_p}(2)$.

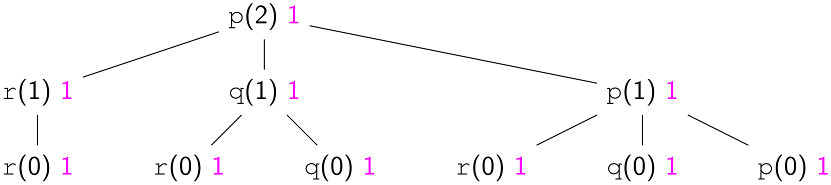
(assume the builtins $>$ /2 and *is*/2 have zero cost)



# Standard Cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

The standard cost of a call $\mathtt{p}(2)$ (in number of resolution steps): $\mathtt{C}_{\mathtt{p}}(2) = 10$.
(also: $\mathtt{C}_{\mathtt{r}}(1) = 2$ and $\mathtt{C}_{\mathtt{q}}(1) = 3$).



# Standard Cost Relations Framework: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Cost relations $\quad n = size(X) = X$ (actual value of $X$)

Standard cost of `p`:

$$\begin{aligned} \mathtt{C_p}(0) &= 1 \\ \mathtt{C_p}(n) &= 1 + \mathtt{C_r}(n-1) + \mathtt{C_q}(n-1) + \mathtt{C_p}(n-1) \quad \text{if } n > 0 \end{aligned}$$

Standard cost of `q`:

$$\begin{aligned} \mathtt{C_q}(0) &= 1 \\ \mathtt{C_q}(n) &= 1 + \mathtt{C_r}(n-1) + \mathtt{C_q}(n-1) \quad \text{if } n > 0 \end{aligned}$$

Standard cost of `r`:

$$\begin{aligned} \mathtt{C_r}(0) &= 1 \\ \mathtt{C_r}(n) &= 1 + \mathtt{C_r}(n-1) \quad \text{if } n > 0 \end{aligned}$$

# Standard Cost Relations Framework: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Cost relations $\quad n = size(X) = X$ (actual value of $X$)

Standard cost of p:

$$\begin{aligned} \mathtt{C}_{\mathtt{p}}(0) &= 1 \\ \mathtt{C}_{\mathtt{p}}(n) &= 1 + \mathtt{C}_{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

Standard cost of q:

$$\begin{aligned} \mathtt{C}_{\mathtt{q}}(0) &= 1 \\ \mathtt{C}_{\mathtt{q}}(n) &= 1 + \mathtt{C}_{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

Standard cost of r $\rightarrow$ closed-form: $\mathtt{C}_{\mathtt{r}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}_{\mathtt{r}}(0) &= 1 \\ \mathtt{C}_{\mathtt{r}}(n) &= 1 + \mathtt{C}_{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

# Standard Cost Relations Framework: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Cost relations $\quad n = size(X) = X$ (actual value of $X$)

Standard cost of $\mathtt{p}$:

$$\begin{aligned} \mathtt{C}_{\mathtt{p}}(0) &= 1 \\ \mathtt{C}_{\mathtt{p}}(n) &= 1 + \mathtt{C}_{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

Standard cost of $\mathtt{q}$:

$$\begin{aligned} \mathtt{C}_{\mathtt{q}}(0) &= 1 \\ \mathtt{C}_{\mathtt{q}}(n) &= 1 + n + \mathtt{C}_{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

Standard cost of $\mathtt{r}$ $\rightarrow$ closed-form: $\mathtt{C}_{\mathtt{r}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}_{\mathtt{r}}(0) &= 1 \\ \mathtt{C}_{\mathtt{r}}(n) &= 1 + \mathtt{C}_{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

# Standard Cost Relations Framework: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Cost relations $\quad n = size(X) = X$ (actual value of $X$)

Standard cost of `p`:

$$\begin{aligned} \mathtt{C}_{\mathtt{p}}(0) = &\quad 1 \\ \mathtt{C}_{\mathtt{p}}(n) = &\quad 1 + \mathtt{C}_{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

Standard cost of `q` $\rightarrow$ closed form: $\mathtt{C}_{\mathtt{q}}(n) = \frac{1}{2}\, n^2 + \frac{3}{2}\, n + 2$ for $n \geq 0$.

$$\begin{aligned} \mathtt{C}_{\mathtt{q}}(0) = &\quad 1 \\ \mathtt{C}_{\mathtt{q}}(n) = &\quad 1 + n + \mathtt{C}_{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

Standard cost of `r` $\rightarrow$ closed-form: $\mathtt{C}_{\mathtt{r}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}_{\mathtt{r}}(0) = &\quad 1 \\ \mathtt{C}_{\mathtt{r}}(n) = &\quad 1 + \mathtt{C}_{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

# Standard Cost Relations Framework: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Cost relations $n = size(X) = X$ (actual value of $X$)

Standard cost of `p`:

$$\mathrm{C_p}(0) = 1$$
$$\mathrm{C_p}(n) = 1 + n + \frac{1}{2}(n-1)^2 + \frac{3}{2}(n-1) + 2 + \mathrm{C_p}(n-1) \quad \text{if } n > 0$$

Standard cost of `q` $\rightarrow$ closed form: $\mathrm{C_q}(n) = \frac{1}{2}n^2 + \frac{3}{2}n + 2$ for $n \geq 0$.

$$\mathrm{C_q}(0) = 1$$
$$\mathrm{C_q}(n) = 1 + n + \mathrm{C_q}(n-1) \quad \text{if } n > 0$$

Standard cost of `r` $\rightarrow$ closed-form: $\mathrm{C_r}(n) = n + 1$, for $n \geq 0$.

$$\mathrm{C_r}(0) = 1$$
$$\mathrm{C_r}(n) = 1 + \mathrm{C_r}(n-1) \quad \text{if } n > 0$$

# Standard Cost Relations Framework: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Cost relations $n = size(X) = X$ (actual value of $X$)

Standard cost of $\mathtt{p}$ → closed form: $\mathtt{C}_{\mathtt{p}}(n) = \frac{1}{6}n^3 + n^2 + \frac{17}{6}n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}_{\mathtt{p}}(0) = & \ 1 \\ \mathtt{C}_{\mathtt{p}}(n) = & \ 1 + n + \frac{1}{2}\ (n-1)^2 + \frac{3}{2}\ (n-1) + 2 + \mathtt{C}_{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

Standard cost of $\mathtt{q}$ → closed form: $\mathtt{C}_{\mathtt{q}}(n) = \frac{1}{2}\ n^2 + \frac{3}{2}\ n + 2$ for $n \geq 0$.

$$\begin{aligned} \mathtt{C}_{\mathtt{q}}(0) = & \ 1 \\ \mathtt{C}_{\mathtt{q}}(n) = & \ 1 + n + \mathtt{C}_{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

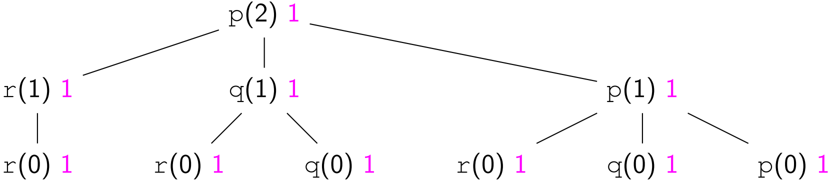
Standard cost of $\mathtt{r}$ → closed-form: $\mathtt{C}_{\mathtt{r}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}_{\mathtt{r}}(0) = & \ 1 \\ \mathtt{C}_{\mathtt{r}}(n) = & \ 1 + \mathtt{C}_{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

# Accumulated-cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

We want to know how the standard/total cost of p is distributed between the predicates of the program.



# Accumulated-cost: Intuition

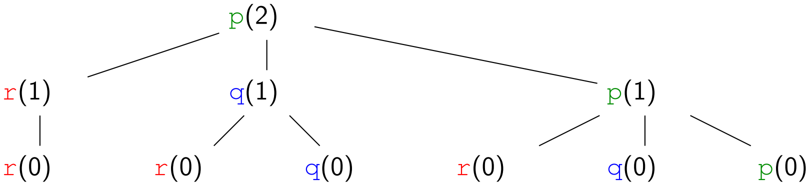
```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:

$$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$$

We declare that predicates p, q, and r are cost centers.

Cost centers are user-defined program points (predicates, in our case) to which execution costs are assigned during the execution of a program.



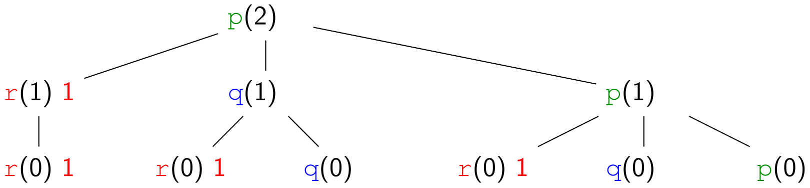
# Accumulated-cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

The cost of a call $\mathtt{p}(2)$ accumulated in cost center $\mathtt{r}$, denoted $\mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(2)$

Is the sum of the resolution steps that are descendant (in the call stack) of $\mathtt{p}(2)$, and whose closest ancestor in the call stack that is a cost center, is $\mathtt{r}$



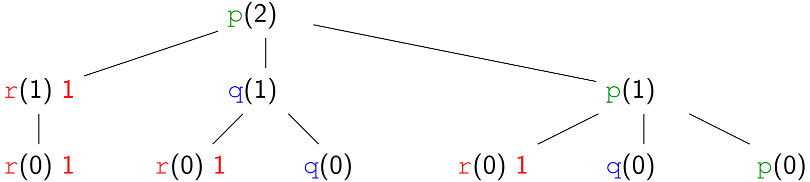
# Accumulated-cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

The cost of a call $\mathtt{p}(2)$ accumulated in cost center $\mathtt{r} \rightarrow \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(2) = 4$

Is the sum of the resolution steps that are descendant (in the call stack) of $\mathtt{p}(2)$, and whose closest ancestor in the call stack that is a cost center, is $\mathtt{r}$



# Accumulated-cost: Intuition

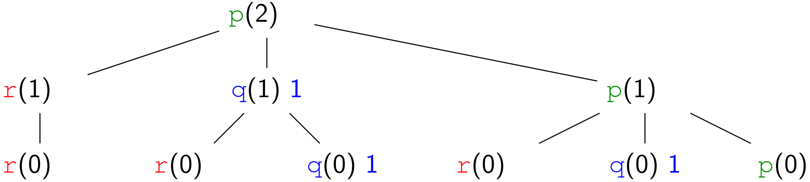
```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:

$$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$$

The cost of a call $\mathtt{p}(2)$ accumulated in cost center $\mathtt{q}$, denoted $\mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(2)$

Is the sum of the resolution steps that are descendant (in the call stack) of $\mathtt{p}(2)$, and whose closest ancestor in the call stack that is a cost center, is $\mathtt{q}$



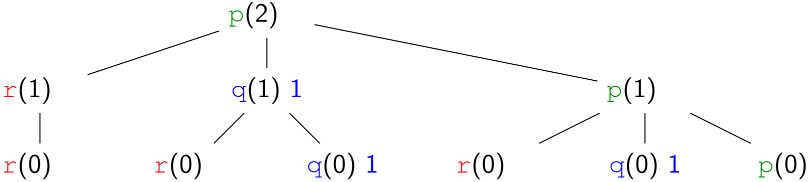
# Accumulated-cost: Intuition

```
p(0).
p(X) :- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X) :- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X) :- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

The cost of a call $\mathtt{p}(2)$ accumulated in cost center $\mathtt{q} \rightarrow \mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(2) = 3$

Is the sum of the resolution steps that are descendant (in the call stack) of $\mathtt{p}(2)$, and whose closest ancestor in the call stack that is a cost center, is $\mathtt{q}$



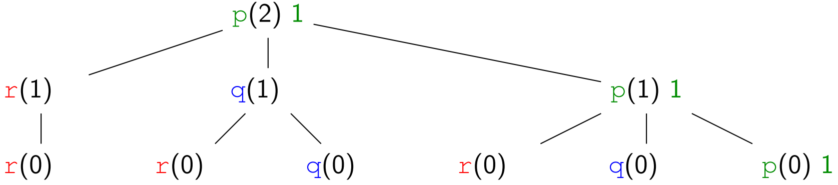
# Accumulated-cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$$

The cost of a call $\mathtt{p}(2)$ accumulated in cost center $\mathtt{p}$, denoted $\mathtt{C}_{\mathtt{p}}^{\mathtt{p}}(2)$

Is the sum of the resolution steps that are descendant (in the call stack) of $\mathtt{p}(2)$, and whose closest ancestor in the call stack that is a cost center, is $\mathtt{p}$



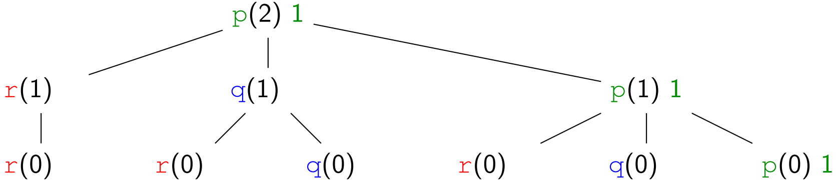
# Accumulated-cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

The cost of a call $\mathtt{p}(2)$ accumulated in cost center $\mathtt{p} \rightarrow \mathtt{C}^{\mathtt{p}}_{\mathtt{p}}(2) = 3$

Is the sum of the resolution steps that are descendant (in the call stack) of $\mathtt{p}(2)$, and whose closest ancestor in the call stack that is a cost center, is $\mathtt{p}$



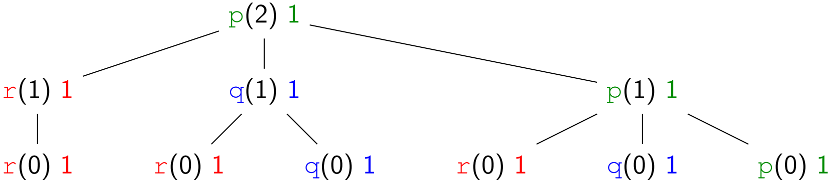
# Accumulated-cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:

$$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$$

$$\begin{array}{ccccccc} \mathtt{C_p}(2) & = & \mathtt{C_p^p}(2) & + & \mathtt{C_p^q}(2) & + & \mathtt{C_p^r}(2) \\ 10 & = & 3 & + & 3 & + & 4 \end{array}$$

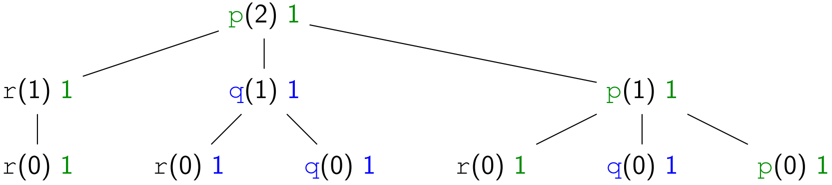


# Accumulated-cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{p, q\}$

We declare that predicates p, q, are cost centers, and r is not.



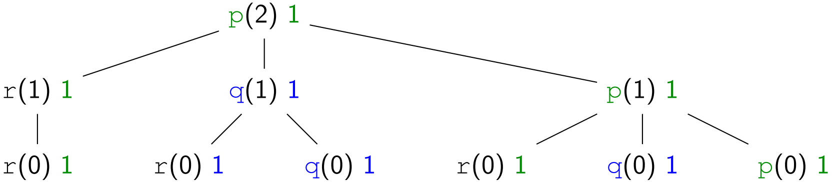
# Accumulated-cost: Intuition

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:

$$\Diamond = \{\mathtt{p}, \mathtt{q}\}$$

$$\begin{array}{ccccc} \mathtt{C_p}(2) & = & \mathtt{C_p^p}(2) & + & \mathtt{C_p^q}(2) \\ 10 & = & 6 & + & 4 \end{array}$$

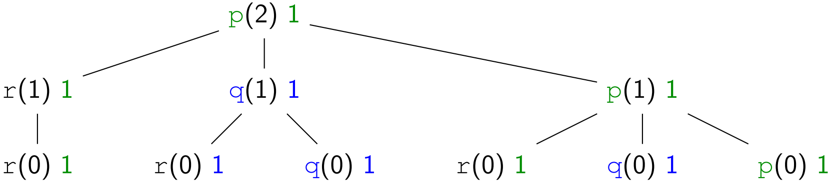


# Accumulated-cost: Definition

## Definition: Accumulated Cost

The cost of a (single) call $\mathtt{p}(n)$ accumulated in cost center $\mathtt{q}$, denoted $\mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(n)$:

- Is the sum of the costs of all the computations that are descendants (in the call stack) of the call $\mathtt{p}(n)$, and are under the scope of any call to $\mathtt{q}$.
- We say that a computation is under the scope of a call to cost center $\mathtt{q}$, if the closest ancestor of such computation in the call stack that is a cost center, is $\mathtt{q}$.
- Expresses how much of the standard cost of the call to $\mathtt{p}$ is attributed to $\mathtt{q}$.



# Cost Relations for Accumulated-costs in Cost Center r

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations

$n = size(X) = X$ (actual value of $X$)

The cost of p accumulated in r:

$$\mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(0) = 0$$
$$\mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(n) = 0 + \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0$$

The cost of q accumulated in r

$$\mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(0) = 0$$
$$\mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n) = 0 + \ + \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0$$

The cost of r accumulated in r

$$\mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(0) = 1$$
$$\mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n) = 1 + \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0$$

# Cost Relations for Accumulated-costs in Cost Center r

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

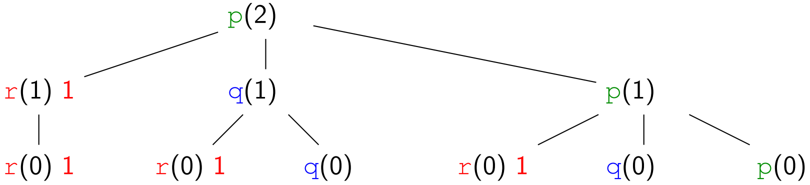
Cost relations $\qquad n = \mathit{size}(X) = X$ (actual value of $X$)

The cost of p accumulated in r:

$$\begin{aligned} \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(0) &= \quad 0 \\ \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(n) &= \quad 0 + \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

---

E.g. ($n = 2$): $\mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(2) = \quad 0 + \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(1) + \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(1) = \quad 0 + 2 + 1 + 1 = 4$



# Cost Relations for Accumulated-costs in Cost Center r

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $\qquad n = size(X) = X$ (actual value of $X$)

The cost of p accumulated in r:

$$\mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(0) = 0$$
$$\mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(n) = 0 + \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0$$

The cost of q accumulated in r:

$$\mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(0) = 0$$
$$\mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n) = 0 + \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0$$

The cost of r accumulated in r:

$$\mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(0) = 1$$
$$\mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n) = 1 + \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0$$

# Cost Relations for Accumulated-costs in Cost Center $\texttt{r}$

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:

$$\lozenge = \{\texttt{p}, \texttt{q}, \texttt{r}\}$$

Cost relations $\qquad n = size(X) = X$ (actual value of $X$)

The cost of $\texttt{p}$ accumulated in $\texttt{r}$:

$$\begin{aligned} \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(0) &= 0 \\ \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(n) &= 0 + \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n-1) + \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n-1) + \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\texttt{q}$ accumulated in $\texttt{r}$:

$$\begin{aligned} \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(0) &= 0 \\ \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n) &= 0 + \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n-1) + \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\texttt{r}$ accumulated in $\texttt{r}$ $\rightarrow$ closed form: $\mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(0) &= 1 \\ \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n) &= 1 + \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

# Cost Relations for Accumulated-costs in Cost Center $\mathtt{r}$

```
p(0).
p(X) :- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X) :- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X) :- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $\qquad n = size(X) = X$ (actual value of $X$)

The cost of $\mathtt{p}$ accumulated in $\mathtt{r}$:

$$\mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(0) = 0$$
$$\mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(n) = 0 + \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n-1) + \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n-1) + \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(n-1) \quad \text{if } n > 0$$

The cost of $\mathtt{q}$ accumulated in $\mathtt{r}$:

$$\mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(0) = 0$$
$$\mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n) = 0 + n + \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n-1) \quad \text{if } n > 0$$

The cost of $\mathtt{r}$ accumulated in $\mathtt{r}$ $\rightarrow$ closed form: $\mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n) = n + 1$, for $n \geq 0$.

$$\mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(0) = 1$$
$$\mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n) = 1 + \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n-1) \quad \text{if } n > 0$$

# Cost Relations for Accumulated-costs in Cost Center $\mathtt{r}$

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $\qquad n = size(X) = X$ (actual value of $X$)

The cost of $\mathtt{p}$ accumulated in $\mathtt{r}$:

$$\begin{aligned} \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(0) &= \quad 0 \\ \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(n) &= \quad 0 + \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n-1) + \mathtt{C}_{\mathtt{p}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\mathtt{q}$ accumulated in $\mathtt{r}$ $\rightarrow$ $\mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n) = \frac{1}{2}n^2 + \frac{1}{2}n$, for $n \geq 0$

$$\begin{aligned} \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(0) &= \quad 0 \\ \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n) &= \quad 0 + n + \mathtt{C}_{\mathtt{q}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\mathtt{r}$ accumulated in $\mathtt{r}$ $\rightarrow$ closed form: $\mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(0) &= \quad 1 \\ \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n) &= \quad 1 + \mathtt{C}_{\mathtt{r}}^{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

# Cost Relations for Accumulated-costs in Cost Center $\mathtt{r}$

```
p(0).
p(X) :- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X) :- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X) :- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $\qquad n = size(X) = X$ (actual value of $X$)

The cost of $\mathtt{p}$ accumulated in $\mathtt{r}$:

$$\begin{aligned} \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(0) &= 0 \\ \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(n) &= 0 + n + \tfrac{1}{2}(n-1)^2 + \tfrac{1}{2}(n-1) + \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\mathtt{q}$ accumulated in $\mathtt{r}$ $\rightarrow$ $\mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n) = \frac{1}{2}n^2 + \frac{1}{2}n$, for $n \geq 0$

$$\begin{aligned} \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(0) &= 0 \\ \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n) &= 0 + n + \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\mathtt{r}$ accumulated in $\mathtt{r}$ $\rightarrow$ closed form: $\mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(0) &= 1 \\ \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n) &= 1 + \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

# Cost Relations for Accumulated-costs in Cost Center $\mathtt{r}$

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $n = size(X) = X$ (actual value of $X$)

The cost of $\mathtt{p}$ accumulated in $\mathtt{r} \rightarrow \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(n) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(0) &= 0 \\ \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(n) &= 0 + n + \tfrac{1}{2}(n-1)^2 + \tfrac{1}{2}(n-1) + \mathtt{C}^{\mathtt{r}}_{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\mathtt{q}$ accumulated in $\mathtt{r} \rightarrow \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n) = \frac{1}{2}n^2 + \frac{1}{2}n$, for $n \geq 0$

$$\begin{aligned} \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(0) &= 0 \\ \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n) &= 0 + n + \mathtt{C}^{\mathtt{r}}_{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\mathtt{r}$ accumulated in $\mathtt{r} \rightarrow$ closed form: $\mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(0) &= 1 \\ \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n) &= 1 + \mathtt{C}^{\mathtt{r}}_{\mathtt{r}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

# Cost Relations for Accumulated-costs in Cost Center $\mathtt{q}$

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $\qquad n = size(X) = X$ (actual value of $X$)

The cost of $\mathtt{p}$ accumulated in $\mathtt{q}$:

$$\mathtt{C}^{\mathtt{q}}_{\mathtt{p}}(0) = 0$$
$$\mathtt{C}^{\mathtt{q}}_{\mathtt{p}}(n) = 0 + \mathtt{C}^{\mathtt{q}}_{\mathtt{r}}(n-1) + \mathtt{C}^{\mathtt{q}}_{\mathtt{q}}(n-1) + \mathtt{C}^{\mathtt{q}}_{\mathtt{p}}(n-1) \quad \text{if } n > 0$$

The cost of $\mathtt{q}$ accumulated in $\mathtt{q}$:

$$\mathtt{C}^{\mathtt{q}}_{\mathtt{q}}(0) = 1$$
$$\mathtt{C}^{\mathtt{q}}_{\mathtt{q}}(n) = 1 + \mathtt{C}^{\mathtt{q}}_{\mathtt{r}}(n-1) + \mathtt{C}^{\mathtt{q}}_{\mathtt{q}}(n-1) \quad \text{if } n > 0$$

The cost of $\mathtt{r}$ accumulated in $\mathtt{q}$:

$$\mathtt{C}^{\mathtt{q}}_{\mathtt{r}}(0) = 0$$
$$\mathtt{C}^{\mathtt{q}}_{\mathtt{r}}(n) = 0 + \mathtt{C}^{\mathtt{q}}_{\mathtt{r}}(n-1) \quad \text{if } n > 0$$

# Cost Relations for Accumulated-costs in Cost Center q

```
p(0).
p(X) :- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X) :- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X) :- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{p, q, r\}$

Cost relations $\qquad n = size(X) = X$ (actual value of $X$)

The cost of p accumulated in q:

$$
\begin{aligned}
C_p^q(0) &= \quad 0 \\
C_p^q(n) &= \quad 0 + C_r^q(n-1) + C_q^q(n-1) + C_p^q(n-1) \quad \text{if } n > 0
\end{aligned}
$$

The cost of q accumulated in q:

$$
\begin{aligned}
C_q^q(0) &= \quad 1 \\
C_q^q(n) &= \quad 1 + C_r^q(n-1) + C_q^q(n-1) \quad \text{if } n > 0
\end{aligned}
$$

The cost of r accumulated in q $\rightarrow C_r^q(n) = 0$, for $n \geq 0$.

$\forall r, q \in \Diamond$, if $r \not\rightarrow_\alpha^\star q$ then $C_r^q(\bar{x}) = 0$ (Lemma 3)

# Cost Relations for Accumulated-costs in Cost Center q

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $\qquad n = size(X) = X$ (actual value of $X$)

The cost of p accumulated in q:

$$\mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(0) = 0$$
$$\mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(n) = 0 + \mathtt{C}_{\mathtt{r}}^{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(n-1) \quad \text{if } n > 0$$

The cost of q accumulated in q $\rightarrow \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n) = n + 1$, for $n \geq 0$.

$$\mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(0) = 1$$
$$\mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n) = 1 + \mathtt{C}_{\mathtt{r}}^{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n-1) \quad \text{if } n > 0$$

The cost of r accumulated in q $\rightarrow \mathtt{C}_{\mathtt{r}}^{\mathtt{q}}(n) = 0$, for $n \geq 0$.

$\forall \mathtt{r}, \mathtt{q} \in \lozenge$, if $\mathtt{r} \not\rightsquigarrow_{\alpha}^{\star} \mathtt{q}$ then $\mathtt{C}_{\mathtt{r}}^{\mathtt{q}}(\bar{\mathtt{x}}) = 0$ (Lemma 3)

# Cost Relations for Accumulated-costs in Cost Center $\mathtt{q}$

```
p(0).
p(X) :- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X) :- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X) :- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $\quad n = size(X) = X$ (actual value of $X$)

The cost of $\mathtt{p}$ accumulated in $\mathtt{q} \rightarrow \mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(n) = \frac{1}{2}\ n^2 + \frac{1}{2}\ n$.

$$\begin{aligned} \mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(0) &= \quad 0 \\ \mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(n) &= \quad 0 + \mathtt{C}_{\mathtt{r}}^{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\mathtt{q}$ accumulated in $\mathtt{q} \rightarrow \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(0) &= \quad 1 \\ \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n) &= \quad 1 + \mathtt{C}_{\mathtt{r}}^{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of $\mathtt{r}$ accumulated in $\mathtt{q} \rightarrow \mathtt{C}_{\mathtt{r}}^{\mathtt{q}}(n) = 0$, for $n \geq 0$.

$\forall \mathtt{r}, \mathtt{q} \in \lozenge$, if $\mathtt{r} \not\rightsquigarrow_{\alpha}^{\star} \mathtt{q}$ then $\mathtt{C}_{\mathtt{r}}^{\mathtt{q}}(\bar{\mathtt{x}}) = 0$ (Lemma 3)

# Cost Relations for Accumulated-costs in Cost Center p

```
p(0).
p(X) :- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X) :- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X) :- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\lozenge = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $\qquad n = size(X) = X$ (actual value of $X$)

The cost of p accumulated in p:

$$\mathtt{C}_{\mathtt{p}}^{\mathtt{p}}(0) = 1$$
$$\mathtt{C}_{\mathtt{p}}^{\mathtt{p}}(n) = 1 + \mathtt{C}_{\mathtt{r}}^{\mathtt{p}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{p}}(n-1) + \mathtt{C}_{\mathtt{p}}^{\mathtt{p}}(n-1) \quad \text{if } n > 0$$

$\mathtt{C}_{\mathtt{q}}^{\mathtt{p}}(n) = 0$ (by Lemma 3, since $\mathtt{q} \not\rightarrowtail_{\alpha}^{\star} \mathtt{p}$).

$\mathtt{C}_{\mathtt{r}}^{\mathtt{p}}(n) = 0$ (by Lemma 3, since $\mathtt{r} \not\rightarrowtail_{\alpha}^{\star} \mathtt{p}$).

# Cost Relations for Accumulated-costs in Cost Center $\mathtt{p}$

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations

$n = size(X) = X$ (actual value of $X$)

The cost of $\mathtt{p}$ accumulated in $\mathtt{p}$:

$$\begin{aligned} \mathtt{C}_{\mathtt{p}}^{\mathtt{p}}(0) &= 1 \\ \mathtt{C}_{\mathtt{p}}^{\mathtt{p}}(n) &= 1 + \mathtt{C}_{\mathtt{p}}^{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

$\mathtt{C}_{\mathtt{q}}^{\mathtt{p}}(n) = 0$ (by Lemma 3, since $\mathtt{q} \not\rightsquigarrow_{\alpha}^{\star} \mathtt{p}$).

$\mathtt{C}_{\mathtt{r}}^{\mathtt{p}}(n) = 0$ (by Lemma 3, since $\mathtt{r} \not\rightsquigarrow_{\alpha}^{\star} \mathtt{p}$).

# Cost Relations for Accumulated-costs in Cost Center $\mathtt{p}$

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:
$\Diamond = \{\mathtt{p}, \mathtt{q}, \mathtt{r}\}$

Cost relations $\quad n = size(X) = X$ (actual value of $X$)

The cost of $\mathtt{p}$ accumulated in $\mathtt{p} \rightarrow \mathtt{C}^{\mathtt{p}}_{\mathtt{p}}(n) = n + 1$, for $n \geq 0$.

$$\begin{aligned} \mathtt{C}^{\mathtt{p}}_{\mathtt{p}}(0) &= \quad 1 \\ \mathtt{C}^{\mathtt{p}}_{\mathtt{p}}(n) &= \quad 1 + \mathtt{C}^{\mathtt{p}}_{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

$\mathtt{C}^{\mathtt{p}}_{\mathtt{q}}(n) = 0$ (by Lemma 3, since $\mathtt{q} \not\rightsquigarrow^{\star}_{\alpha} \mathtt{p}$).

$\mathtt{C}^{\mathtt{p}}_{\mathtt{r}}(n) = 0$ (by Lemma 3, since $\mathtt{r} \not\rightsquigarrow^{\star}_{\alpha} \mathtt{p}$).

# Need for Tracking the "Environment:" Example

```
p(0).
p(X) :- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X) :- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X) :- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:

$$\Diamond = \{\mathtt{p}, \mathtt{q}\}$$

The cost of p accumulated in q:

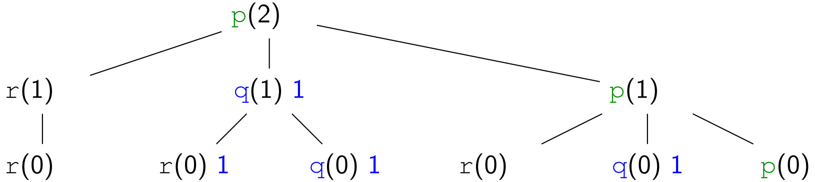
$$\mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(0) = 0$$

$$\mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(n) = 0 + \mathtt{C}_{\mathtt{r},0}^{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{p}}^{\mathtt{q}}(n-1) \quad \text{if } n > 0$$

The cost of q accumulated in q:

$$\mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(0) = 1$$

$$\mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n) = 1 + \mathtt{C}_{\mathtt{r},1}^{\mathtt{q}}(n-1) + \mathtt{C}_{\mathtt{q}}^{\mathtt{q}}(n-1) \quad \text{if } n > 0$$




We have two versions for the cost of r accumulated in q:

# Need for Tracking the "Environment:" Example

```
p(0).
p(X):- X > 0, Y is X - 1, r(Y), q(Y), p(Y).

q(0).
q(X):- X > 0, Y is X - 1, r(Y), q(Y).

r(0).
r(X):- X > 0, Y is X - 1, r(Y).
```

Set of cost centers:

$$\Diamond = \{\mathtt{p}, \mathtt{q}\}$$

The cost of p accumulated in q:

$$\begin{aligned} \mathtt{C}^{\mathtt{q}}_{\mathtt{p}}(0) &= 0 \\ \mathtt{C}^{\mathtt{q}}_{\mathtt{p}}(n) &= 0 + \mathtt{C}^{\mathtt{q}}_{\mathtt{r},0}(n-1) + \mathtt{C}^{\mathtt{q}}_{\mathtt{q}}(n-1) + \mathtt{C}^{\mathtt{q}}_{\mathtt{p}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

The cost of q accumulated in q:

$$\begin{aligned} \mathtt{C}^{\mathtt{q}}_{\mathtt{q}}(0) &= 1 \\ \mathtt{C}^{\mathtt{q}}_{\mathtt{q}}(n) &= 1 + \mathtt{C}^{\mathtt{q}}_{\mathtt{r},1}(n-1) + \mathtt{C}^{\mathtt{q}}_{\mathtt{q}}(n-1) \quad \text{if } n > 0 \end{aligned}$$

---

We have two versions for the cost of r accumulated in q:

Under the scope of q

$$\boxed{\begin{aligned} \mathtt{C}^{\mathtt{q}}_{\mathtt{r},1}(0) &= 1 \\ \mathtt{C}^{\mathtt{q}}_{\mathtt{r},1}(n) &= 1 + \mathtt{C}^{\mathtt{q}}_{\mathtt{r},1}(n-1) \quad \text{if } n > 0 \end{aligned}}$$

NOT under the scope of q

$$\boxed{\begin{aligned} \mathtt{C}^{\mathtt{q}}_{\mathtt{r},0}(0) &= 0 \\ \mathtt{C}^{\mathtt{q}}_{\mathtt{r},0}(n) &= 0 + \mathtt{C}^{\mathtt{q}}_{\mathtt{r},0}(n-1) \quad \text{if } n > 0 \end{aligned}}$$

# Our Extended Cost Relations for Accumulated-cost

The standard cost of a clause

$$C \equiv \mathrm{p}(\bar{x}) :- \mathrm{q}_1(\bar{x}_1), \ldots, \mathrm{q}_n(\bar{x}_n)$$

for a (single) call to $\mathrm{p}$:

$$\mathrm{C}_{\mathrm{p}}(\bar{\mathrm{x}}) \quad = \varphi(\mathrm{p}(\bar{\mathrm{x}})) + \sum_{i=1}^{lim(C,\bar{\mathrm{x}})} sols_i \times \mathrm{C}_{\mathrm{q}_i}(\bar{\mathrm{x}}_i)$$

E.g., for resolutions steps $\rightarrow \varphi(\mathrm{p}(\bar{\mathrm{x}})) = 1$.

- $lim(C, \bar{\mathrm{x}}) \stackrel{\text{def}}{=}$ index of the last body literal that is called in the execution of $C$.
- $sols_i \stackrel{\text{def}}{=}$ product of the number of solutions produced by the ancestor literals of $\mathrm{q}_i(\bar{x}_i)$ in the clause body:

$$sols_i = \prod_{j=1}^{i-1} s_{pred}(\mathrm{q}_j(\bar{x}_j))$$

$$s_{pred}(\mathrm{q}_j(\bar{x}_j)) \stackrel{\text{def}}{=} \text{number of solutions produced by } \mathrm{q}_j(\bar{x}_j)$$

The cost of a body literal $\mathrm{q}_i(\bar{x}_i)$ is obtained from the costs of all clauses applicable to it that are executed, by using an aggregation operator $\bigodot$

# Our Extended Cost Relations for Accumulated-cost

The accumulated cost of a clause

$$C \equiv \mathrm{p}(\bar{x}) :- \mathrm{q}_1(\bar{x}_1), \ldots, \mathrm{q}_n(\bar{x}_n)$$

for a (single) call to p:

$$\mathrm{C}^{\mathrm{c}}_{\mathrm{p},e}(\bar{\mathrm{x}}) \quad = B_{\varphi}(\mathrm{p}, \mathrm{c}, e) \times \varphi(\mathrm{p}(\bar{\mathrm{x}})) + \sum_{i=1}^{lim(C,\bar{\mathrm{x}})} sols_i \times \mathrm{C}^{\mathrm{c}}_{\mathrm{q}_i,e'}(\bar{\mathrm{x}}_i) \times B(\mathrm{p}, \mathrm{c}, e, \mathrm{q}_i)$$

E.g., for resolutions steps $\rightarrow \varphi(\mathrm{p}(\bar{\mathrm{x}})) = 1$.

- $lim(C, \bar{\mathrm{x}}) \stackrel{\text{def}}{=}$ index of the last body literal that is called in the execution of $C$.
- $sols_i \stackrel{\text{def}}{=}$ product of the number of solutions produced by the ancestor literals of $\mathrm{q}_i(\bar{x}_i)$ in the clause body:

$$sols_i = \prod_{j=1}^{i-1} s_{pred}(\mathrm{q}_j(\bar{x}_j))$$

$$s_{pred}(\mathrm{q}_j(\bar{x}_j)) \stackrel{\text{def}}{=} \text{number of solutions produced by } \mathrm{q}_j(\bar{x}_j)$$

The cost of a body literal $\mathrm{q}_i(\bar{x}_i)$ is obtained from the costs of all clauses applicable to it that are executed, by using an aggregation operator $\bigodot$

# Our Extended Cost Relations for Accumulated-cost

The accumulated cost of a clause

$$C \equiv \mathrm{p}(\bar{x}) :- \mathrm{q}_1(\bar{x}_1), \ldots, \mathrm{q}_n(\bar{x}_n)$$

for a (single) call to $\mathrm{p}$:

$$\mathrm{C}^{\mathrm{c}}_{\mathrm{p},e}(\bar{\mathrm{x}}) = B_\varphi(\mathrm{p},\mathrm{c},e) \times \varphi(\mathrm{p}(\bar{\mathrm{x}})) + \sum_{i=1}^{lim(C,\bar{\mathrm{x}})} sols_i \times \mathrm{C}^{\mathrm{c}}_{\mathrm{q}_i,e'}(\bar{\mathrm{x}}_i) \times B(\mathrm{p},\mathrm{c},e,\mathrm{q}_i)$$

- The environment $e$ is a Boolean value ($1 \equiv \mathtt{true}$ and $0 \equiv \mathtt{false}$):
  $$e = \begin{cases} 1 & \text{if the call to } \mathrm{p} \text{ is under the scope of cost center } \mathrm{c} \\ 0 & \textit{otherwise} \end{cases}$$
- Boolean functions:
  $B_\varphi(\mathrm{p},\mathrm{c},e)$ is 1 iff "the computation" is under the scope of $\mathrm{c}$.
  $$B_\varphi(\mathrm{p},\mathrm{c},e) \stackrel{\text{def}}{=} (\mathrm{p} = \mathrm{c} \vee (\mathrm{p} \notin \Diamond \wedge e))$$
  $B(p,\mathrm{c},e,q)$ is 1 iff the body literal is under the scope of $\mathrm{c}$, or it may call $\mathrm{c}$.
  $$B(p,\mathrm{c},e,q) \stackrel{\text{def}}{=} B_\varphi(p,\mathrm{c},e) \vee (q \rightsquigarrow^{\star}_{\alpha} c)$$
- $e' = \mathcal{E}(\mathrm{p},\mathrm{c},e,\mathrm{q}_i(\bar{\mathrm{x}}_i))$, and $\mathcal{E}$ is the *environment change* function:
  $\mathcal{E}(\mathrm{p},\mathrm{c},e,\_) \stackrel{\text{def}}{=} (\mathrm{p} = \mathrm{c} \vee (\mathrm{p} \notin \Diamond \wedge e))$

# Our Extended Cost Relations for Accumulated-cost

The accumulated cost of a clause

$$C \equiv \mathrm{p}(\bar{x}) :- \mathrm{q}_1(\bar{x}_1), \ldots, \mathrm{q}_n(\bar{x}_n)$$

for a (single) call to p:

$$\mathrm{C}^{\mathrm{c}}_{\mathrm{p},e}(\bar{\mathrm{x}}) \quad = B_{\varphi}(\mathrm{p}, \mathrm{c}, e) \times \varphi(\mathrm{p}(\bar{\mathrm{x}})) + \sum_{i=1}^{lim(C,\bar{\mathrm{x}})} sols_i \times \mathrm{C}^{\mathrm{c}}_{\mathrm{q}_i,e'}(\bar{\mathrm{x}}_i) \times B(\mathrm{p}, \mathrm{c}, e, \mathrm{q}_i)$$

If a trust assertion gives the cost of p as a function $\Psi(\mathrm{p})(\bar{\mathrm{x}})$, then:
$\mathrm{C}_{\mathrm{p}}(\bar{\mathrm{x}}) = \Psi(\mathrm{p})(\bar{\mathrm{x}})$

# Our Extended Cost Relations for Accumulated-cost

The accumulated cost of a clause

$$C \equiv \mathrm{p}(\bar{x}) :- \mathrm{q}_1(\bar{x}_1), \ldots, \mathrm{q}_n(\bar{x}_n)$$

for a (single) call to p:

$$\mathrm{C}^{\mathrm{c}}_{\mathrm{p},e}(\bar{\mathrm{x}}) \quad = B_{\varphi}(\mathrm{p}, \mathrm{c}, e) \times \varphi(\mathrm{p}(\bar{\mathrm{x}})) + \sum_{i=1}^{lim(C,\bar{\mathrm{x}})} sols_i \times \mathrm{C}^{\mathrm{c}}_{\mathrm{q}_i,e'}(\bar{\mathrm{x}}_i) \times B(\mathrm{p}, \mathrm{c}, e, \mathrm{q}_i)$$

If a trust assertion gives the cost of p as a function $\Psi(\mathrm{p})(\bar{\mathrm{x}})$, then:

$$\mathrm{C}^{\mathrm{c}}_{\mathrm{p},e}(\bar{\mathrm{x}}) = \Psi(\mathrm{p})(\bar{\mathrm{x}}) \times B_{\varphi}(\mathrm{p}, \mathrm{c}, e)$$

# Genericity of our Cost Relations Framework

The cost a clause for a (single) call to $\mathtt{p}$:

$$\mathtt{C}^{\mathtt{c}}_{\mathtt{p},e}(\bar{\mathtt{x}}) \quad = B_{\varphi}(\mathtt{p},\mathtt{c},e) \times \varphi(\mathtt{p}(\bar{\mathtt{x}})) + \sum_{i=1}^{lim(C,\bar{\mathtt{x}})} sols_i \times B(\mathtt{p},\mathtt{c},e,\mathtt{q}_i) \times \mathtt{C}^{\mathtt{c}}_{\mathtt{q}_i,e'}(\bar{\mathtt{x}}_i)$$

- A broad notion of *environment* $e$. E.g., for energy consumption:
  - state of the hardware or the whole system,
  - the last instruction executed (for modeling the *switching cost*), temperature, voltage, cache state, and pipeline state.
- Suitable definitions of the Boolean functions $B_{\varphi}(\mathtt{p},\mathtt{c},e)$ and $B(p,\mathtt{c},e,q)$ to control which terms of the cost relations should be considered.
- $\mathtt{C}^{\mathtt{c}}_{\mathtt{p},e}(\bar{\mathtt{x}}) \stackrel{\text{def}}{=}$ part of $\mathtt{C}_{\mathtt{p}}(\bar{\mathtt{x}})$, performed in an *environment* $e$, that is attributed to *cost center* $\mathtt{c}$ of the program.

# Some Properties of the Accumulated-cost

### Definition of the *calls* relation, $\rightsquigarrow_\alpha$

- $\mathtt{p} \rightsquigarrow_\alpha \mathtt{q}$, iff a literal with predicate symbol $\mathtt{q}$ appears in the body of a clause defining $\mathtt{p}$.
- $\rightsquigarrow_\alpha^\star$ is the reflexive transitive closure of $\rightsquigarrow_\alpha$.
- It is an abstraction (over-approximation) of the concrete "calls" relation, $\rightsquigarrow$.

# Some Properties of the Accumulated-cost

- $\forall \mathrm{p}, \mathrm{c} \in \lozenge$, $\forall e \in \{0, 1\}$, it holds that:
  - $\mathcal{E}(\mathrm{p}, \mathrm{c}, e, \_) \stackrel{\text{def}}{=} (\mathrm{p} = \mathrm{c})$

    (recall that $\mathcal{E}(\mathrm{p}, \mathrm{c}, e, \_) \stackrel{\text{def}}{=} (\mathrm{p} = \mathrm{c} \vee (\mathrm{p} \notin \lozenge \wedge e)))$
  - $B_\varphi(\mathrm{p}, \mathrm{c}, e) \stackrel{\text{def}}{=} (\mathrm{p} = \mathrm{c})$.

    (recall that $B_\varphi(\mathrm{p}, \mathrm{c}, e) \stackrel{\text{def}}{=} (\mathrm{p} = \mathrm{c} \vee (\mathrm{p} \notin \lozenge \wedge e)))$
  - $B(p, \mathrm{c}, e, q) \stackrel{\text{def}}{=} (\mathrm{p} = \mathrm{c}) \vee (q \rightsquigarrow_\alpha^\star c)$.

    (recall that $B(p, \mathrm{c}, e, q) \stackrel{\text{def}}{=} B_\varphi(p, \mathrm{c}, e) \vee (q \rightsquigarrow_\alpha^\star c))$
- This implies that $\forall \mathrm{p}, \mathrm{c} \in \lozenge$ it holds that $\mathrm{C}^{\mathrm{c}}_{\mathrm{p},0}(\bar{\mathrm{x}}) = \mathrm{C}^{\mathrm{c}}_{\mathrm{p},1}(\bar{\mathrm{x}})$.
- Thus, if $\mathrm{p} \in \lozenge$ we omit the environment $e$ and write $\mathrm{C}^{\mathrm{c}}_{\mathrm{p}}(\bar{\mathrm{x}})$.
- (Lemma 3) $\forall \mathrm{p}, \mathrm{c} \in \lozenge$, if $\mathrm{p} \not\rightsquigarrow_\alpha^\star \mathrm{c}$ then $\mathrm{C}^{\mathrm{c}}_{\mathrm{p}}(\bar{\mathrm{x}}) = 0$.
- (Lemma 4) $\forall \mathrm{p} \notin \lozenge, \forall \mathrm{c} \in \lozenge$, if $\mathrm{p} \not\rightsquigarrow_\alpha^\star \mathrm{c}$ then $\mathrm{C}^{\mathrm{c}}_{\mathrm{p},0}(\bar{\mathrm{x}}) = 0$.

# Usefulness of the Accumulated Cost

Consider the following program, where predicates $p$, $q$ and $r$ are cost centers.

```
p(X, Y, Z):- X > 0, q(X, Y, Z1), Z is Z1 * 2.

q(0, _, 0).
q(X, Y, Z) :- r(Y, Y1), X1 is X - 1, q(X1, Y, Z1), Z is Z1 + Y1.

r(0, 0).
r(X, Y):- X1 is X - 1, r(X1, Y1), Y is Y1 + X.
```

Standard Cost

- $\boxed{\mathcal{C}_p(x,y) = y * x + 2 * x + 2}$
- $\boxed{\mathcal{C}_q(x,y) = y * x + 2 * x + 1}$
- $\boxed{\mathcal{C}_r(x,y) = x + 1}$

Accumulated Cost

- $\boxed{\mathcal{C}_p^p(x,y) = 1}$
- $\boxed{\mathcal{C}_p^q(x,y) = x}$
- $\boxed{\mathcal{C}_p^r(x,y) = x * y}$

- $\mathcal{C}_r(x,y) = x + 1 \rightarrow$ *r is not costly by itself.*
- $\mathcal{C}_p^r(x,y) = x * y \rightarrow$ *r is responsible for the non-linear complexity of p.*

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Consider the following program, where predicates $p$, $q$ and $r$ are cost centers.

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- $\mathcal{C}_p^r(x,y) = x * y \rightarrow$ *r is responsible for the non-linear complexity of p.*

# Usefulness of the Accumulated Cost

Obvious improvement:
move the call $r(Y, Y1)$ outside the (recursive) definition of the predicate $q$.

```
p(X, Y, Z):- X >= 0, r(Y, Y1) , q(X, Y1, Z1), Z is Z1 * 2.

q(0, _, 0).
q(X, Y, Z) :- X1 is X - 1, q(X1, Y, Z1), Z is Z1 + Y.

r(0, 0).
r(X, Y):- X1 is X - 1, r(X1, Y1), Y is Y1 + X.
```

Standard Cost

- $\boxed{\mathcal{C}_p(x, y) = x + y + 3}$
- $\boxed{\mathcal{C}_q(x, y) = x + 1}$
- $\boxed{\mathcal{C}_r(x, y) = x + 1}$

Accumulated Cost

- $\boxed{\mathcal{C}_p^p(x, y) = 1}$
- $\boxed{\mathcal{C}_p^q(x, y) = x}$
- $\boxed{\mathcal{C}_p^r(x, y) = y}$

# Implementation

- Implementation within CiaoPP, directly as an abstract *domain*.
- The information abstracted at each program point includes the state + non-functional props.
- Cost relations are built *incrementally, in the abstract domain*.
- Features inherited for free:
  - Multivariance: separate equations built for each procedure version.
  - Equations are not built for unreachable parts of the program.
  - Easy combination with other abstract domains (reduced product based), in particular, the new sized types and a novel cardinality analysis.
  - Assertion verification.
  - Etc.

# Accumulated Cost: Experimental Results

| Cost-Centers & Input Sizes | Accumulated Cost UB | Static vs. Dyn | Standard Cost UB | #Calls |
|---|---|---|---|---|
| $variance(n)^*$ | $1$ | 0% | $2n^2$ | $1$ |
| $sq\_diff(m_1, m_2)$ | $n-1$ | 0% | $2m_1 m_2 - 2m_2$ | $n-1$ |
| $mean(u)$ | $2n^2 - n$ | 0% | $2u+1$ | $n$ |
| $is\_prime(n)^*$ | $1$ | 0% | $(n-1)! + n + 3$ | $1$ |
| $fact(m)$ | $n$ | 0% | $m$ | $n$ |
| $mult(u)$ | $(n-1)! + 2$ | 0% | $u+1$ | $(n-1)! + 2$ |
| $app1(n_1, n_2, n_3)^*$ | $n_1$ | 0% | $\mathcal{O}(n_1 n_2 n_3)^\dagger$ | $1$ |
| $app2(m_1, m_2)$ | $n_1 n_2$ | 0% | $m_1 m_2$ | $n_1$ |
| $app3(u)$ | $2n_1 n_2 n_3$ | 0% | $u$ | $n_1 n_2 + n_1$ |
| $dyade(n_1, n_2)^*$ | $n_1$ | 0% | $n_1(n_2+1)$ | $1$ |
| $mult(m)$ | $n_1 n_2$ | 0% | $m$ | $n_1$ |
| $minsort(n)^*$ | $n+1$ | 0% | $\frac{(n+1)^2}{2} + \frac{n+1}{2}$ | $1$ |
| $findmin(m)$ | $\frac{(n+1)^2}{2} + \frac{n-1}{2}$ | 7% | $m$ | $n+1$ |
| $hanoi(n)^*$ | $2^n - 1$ | 0% | $2^{n+1} - 2$ | $1$ |
| $move(m)$ | $2^n - 1$ | 0% | $1$ | $2^n - 1$ |
| $coupled(n)^*$ | $1$ | 0% | $n+2$ | $1$ |
| $p(m)$ | $\frac{n}{2} + \frac{(-1)^n}{4} + \frac{3}{4}$ | 1.2% | $m+1$ | $\frac{n}{2} - \frac{(-1)^n}{4} + \frac{1}{4}$ |
| $q(u)$ | $\frac{n}{2} - \frac{(-1)^n}{4} + \frac{1}{4}$ | 0% | $u+1$ | $\frac{n}{2} + \frac{(-1)^n}{4} - \frac{1}{4}$ |
| $search(n)^*$ | $1$ | 0% | $2n+2$ | $1$ |
| $member(m)$ | $2n+1$ | 0% | $2m+1$ | $2n+1$ |
| $sublist(n_1, n_2)^*$ | $n_2 + 3$ | 5% | $n_1 n_2 + 3n_2 + 2$ | $2$ |
| $append(m)$ | $n_1 n_2 + 2n_2 - 1$ | 40% | $2m-1$ | $n_1 n_2 + 2n_2 - 1$ |

# Experimental Results: Times (milliseconds)

| Cost-Center | Accumulated Cost UB | | Standard Cost UB | Acc Cost/ Std Cost |
|---|---|---|---|---|
| | Cost Relations | Transformation (FLOPS'16) | | |
| *variance** *sq_diff* *mean* | 3283 (-45%) | 6038 | 3066 | 1.07 |
| *isprime** *fact* *mult* | 1245 (-42%) | 2172 | 1231 | 1.01 |
| *app1** *app2* *app3* | 4150 (-34%) | 6328 | 3757 | 1.11 |
| *minsort** *findmin* | 3400 (-29%) | 4845 | 3300 | 1.03 |
| *dyade** *mult* | 3097 (-24%) | 4117 | 2853 | 1.08 |
| *hanoi** *move* | 1605 (-19%) | 1996 | 1376 | 1.16 |
| *coupled** *f* *g* | 2407 (-14%) | 3112 | 1877 | 1.28 |
| *search** *member* | 1079 | N/A | 1071 | 1.00 |
| *sublist** *append* | 3674 | N/A | 3610 | 1.01 |
| **Average** | **2652 (-33%)** | **4125** | **2542** | **1.05** |

# Conclusions

- Novel, general, and flexible framework for setting up cost relations which can be instantiated for performing a wide range of resource usage analyses, including both accumulated cost and standard cost.
- Advantages over our previous work (specific to accumulated cost) based on a program transformation:
  - More general.
  - Can deal with non-deterministic/multiple-solution predicates.
  - More efficient.
  - Implementation based on a direct application of abstract interpretation and integration into CiaoPP → many useful features are inherited for free.
  - Also inherits the capability of analyzing for several resources at the same time.
- Experiments → accurate inference of accumulated cost.
- Static profiling is a *more valuable aid for resource-aware software development* than standard resource usage analysis.

  → identify parts that should be optimized first.
- Our approach can be easily applied to other paradigms:

  → including imperative programs, functional programs, CHR, etc.,

  → by compilation to Horn Clauses (as in our previous work with Java or XC).

# Demo!

Please see examples in the CiaoPP playground.

(`http://play.ciao-lang.org`)

# The Team

- Working specifically in CiaoPP resource analysis:


Pedro López-García


Manuel Hermenegildo


Maximiliano Klemen


Umer Liqat

- CiaoPP overall:


José-Francisco Morales


Nataliia Stulova


Isabel García-Contreras

- Previous main contributors to CiaoPP resource analysis:

Saumya Debray　Nai-wei Lin　Jorge Navas
Alejandro Serrano　Mario Méndez-Lojo　Edison Mera

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Work currently at: IMDEA Software Institute, T.U. Madrid (UPM).

And previously at: U. T. Austin, MCC, U. of Arizona, U. of New Mexico.

Playground at: `http://play.ciao-lang.org`

Thank you!

# Timeline of our Work

**1990** Method for static inference of upper-bound functions on execution cost and data structure sizes [PLDI'90] (building on Wegbreit):

- Techniques for setting up, solving/approximating recurrence relations.
- For Horn-clause programs → used widely as IR for other languages.
- Motivation: task granularity control in automatic parallelization.
- Experimental results (resulting in improved parallel speedups).
- Implementation (leading to CASLOG) but I/O arguments, types, measures, etc. had to be provided by the user.

**1993-1994** **First fully automatic system**, including all auxiliary analyses: **GraCos** (Granularity Control System), implemented within **CiaoPP** [SAS'94, PASCO'94].

- Reducing data size computation overhead. [ICLP'95]
- Further improvements. [JSC'96]
- Precision improved w/determinacy, partial eval. ... [LOPSTR'04, NGC'10]

**1997** **Lower bounds** cost analysis; **divide-and-conquer**. [ILPS'97]

- Lower bounds required developing non-failure (no-exceptions) analysis, guard coverage, ... [ICLP'97, FLOPS'04]
- Also in [ILPS'97]: proposed *non-deterministic recurrence relations*, special for divide-and-conquer programs: looking at sets of computation trees and balancing/bounding node cost (e.g., quadratic bound for qsort).

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**2004** Abstraction carrying code for resources. [PPDP'05, LPAR'04]

**2006** Probabilistic Cost Analysis. [CLEI'06]

**2007** **User-definable** resources. [ICLP'07]

**2007** **Multi-language** support (Java bytecode, C#, FP, CLP) via **Horn clause-based IR**. [LOPSTR'07]

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- Instrumental analyses: sharing/nullity/class [VMCAI'08, PASTE'08] dependence [LCPC'08] shape [CC'08, SAS'02].

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**2012-2014** **Cost analysis as multivariant abstract interpretation**. [TPLP'14]

→ Multivariant, integrated with assertion checking, modular, incremental.

Domain: interval (piece-wise) functions. [ICLP'10, FOPARA'12]

**2013** Using **sized shapes (sized types)**. [ICLP'13]

**2013-2016** **Analysis and verification of Energy**:

- At the ISA level [LOPSTR'13]
- Comparing LLMV and ISA levels [FOPARA'15]
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# Additional slides

# "Classical" Cost Analysis (cost relations): Example

```
nrev([],[]).
nrev([H|L],R) :- nrev(L,R1), app(R1,[H],R).

app([],L,L).
app([H|L],L1,[H|R]) :- app(L,L1,R).
```

- Cost relations for resolution steps $\quad n = \mathit{length}(X)$ (length of list $X$)
- Cost of *nrev*:
$$\begin{aligned} \mathrm{C}_{nrev}(0) &= 1 \\ \mathrm{C}_{nrev}(n) &= 1 + \mathrm{C}_{nrev}(n-1) + \mathrm{C}_{app}(n-1, 1) \quad \text{if } n > 0 \end{aligned}$$
- Cost of *app*:
$$\begin{aligned} \mathrm{C}_{app}(0, m) &= 1 \\ \mathrm{C}_{app}(n, m) &= 1 + \mathrm{C}_{app}(n-1, m) \quad \text{if } n > 0 \end{aligned}$$
- Approach described in [PLDI'90], [ILPS'97] (for lower bounds, nondet relations, balanced costs), [ICLP'07, Bytecode'09] (for user-defined resources).

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- Cost of *app* $\rightarrow$ closed form: $\mathrm{C}_{app}(n,m) = n + 1$ for $n \geq 0$.
  $$\mathrm{C}_{app}(0,m) = 1$$
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