Solving Constrained Horn Clauses by Property Directed Reachability

Arie Gurfinkel

HCVS 2017: 4th Workshop on Horn Clauses for Verification and Synthesis

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Automated Verification

Deductive Verification

- A user provides a program and a verification certificate
  - e.g., inductive invariant, pre- and post-conditions, function summaries, etc.
- A tool automatically checks validity of the certificate
  - this is not easy! (might even be undecidable)
- Verification is manual but machine certified

Algorithmic Verification (My research area)

- A user provides a program and a desired specification
  - e.g., program never writes outside of allocated memory
- A tool automatically checks validity of the specification
  - and generates a verification certificate if the program is correct
  - and generates a counterexample if the program is not correct
- Verification is completely automatic – “push-button”

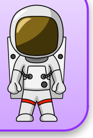
Algorithmic Logic-Based Verification

Safety Properties

Program + Spec

Constrained Horn Clauses

Verification Condition (in Logic)

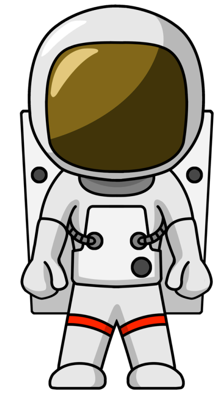
Spacer 

Decision Procedure

Yes

No

Spacer: Solving SMT-constrained CHC



Spacer: a solver for SMT-constrained Horn Clauses

- now part of Z3
  - https://github.com/Z3Prover/z3 since commit `72c4780`
  - use option `fixedpoint.engine=spacer`
- development version at http://bitbucket.org/spacer/code

Supported SMT-Theories

- Best-effort support for many SMT-theories
  - data-structures, bit-vectors, non-linear arithmetic
- Linear Real and Integer Arithmetic
- Quantifier-free theory of arrays
- *Universally quantified theory of arrays + arithmetic (work in progress)*

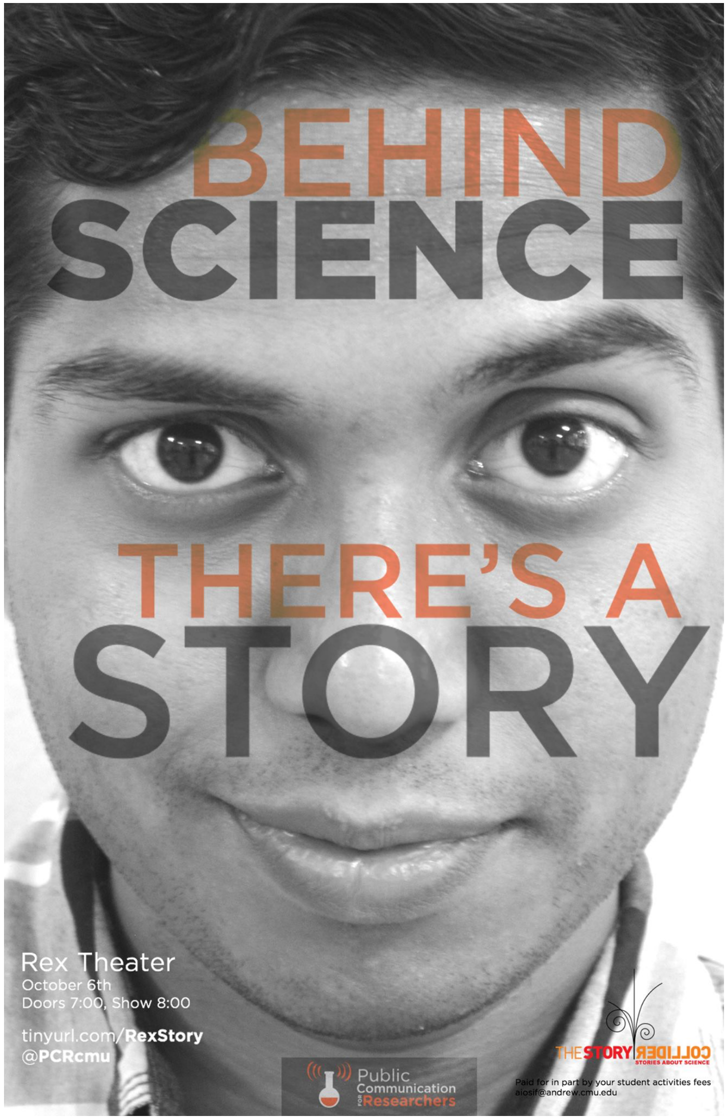
Support for Non-Linear CHC

- for procedure summaries in inter-procedural verification conditions
- for compositional reasoning: abstraction, assume-guarantee, thread modular, etc.

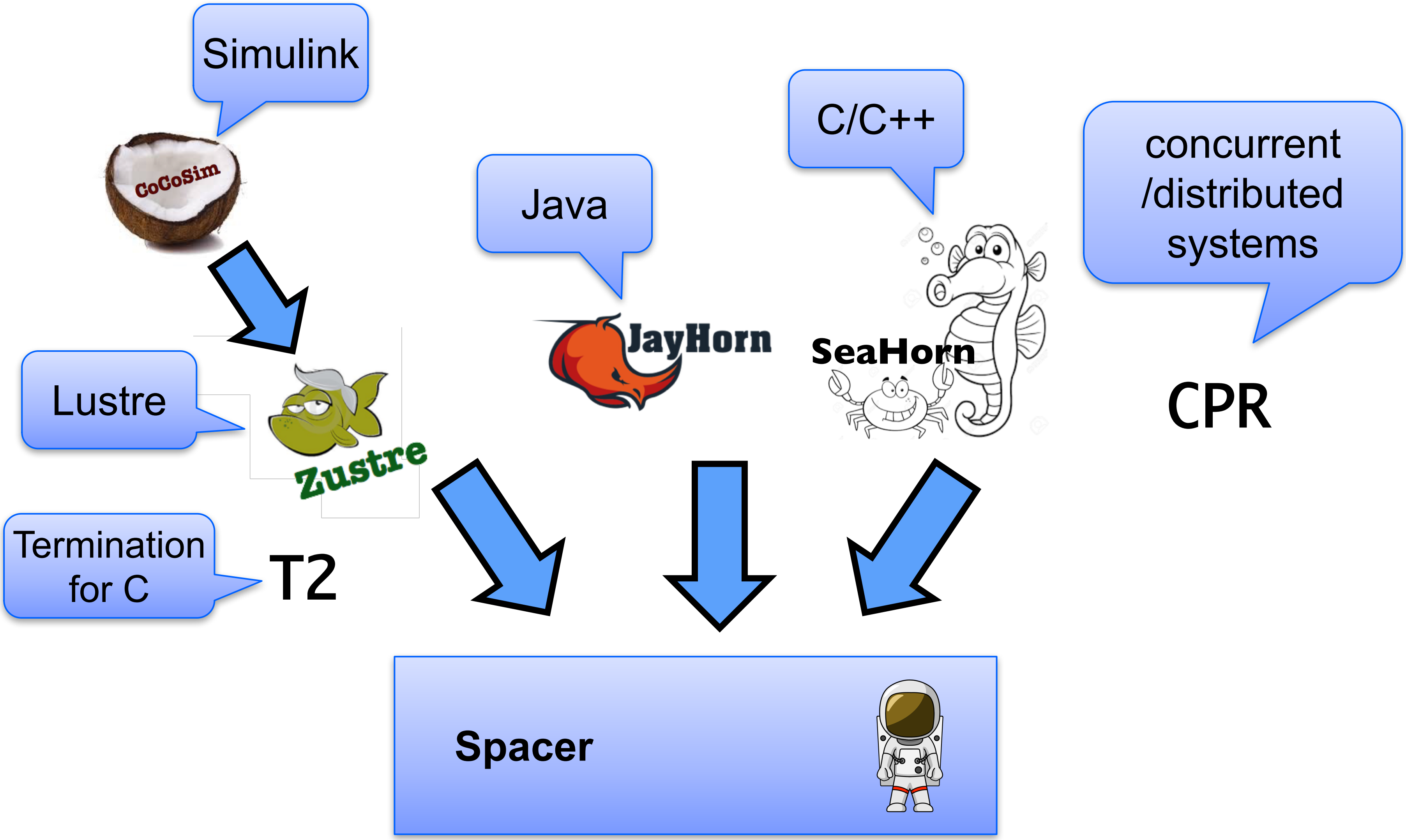
Contributors

Arie Gurfinkel
Anvesh Komuravelli

Nikolaj Bjorner
(Krystof Hoder)
Yakir Vizel
Bernhard Gleiss
Matteo Marescotti



Logic-based Algorithmic Verification



Constrained Horn Clauses (CHC)

A Constrained Horn Clause (CHC) is a FOL formula of the form

$$\forall V \,.\, (\phi \wedge p_1[X_1] \wedge \ldots \wedge p_n[X_n] \rightarrow h[X]),$$

where

- A is a background theory (e.g., Linear Arithmetic, Arrays, Bit-Vectors, or combinations of the above)
- ϕ is a constrained in the background theory A
- $p_1, \ldots, p_n$, h are n-ary predicates
- $p_i[X]$ is an application of a predicate to first-order terms

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CHC Satisfiability

A **model** of a set of clauses Π is an interpretation of each predicate p_i that makes all clauses in Π valid

A set of clauses is **satisfiable** if it has a model, and is unsatisfiable otherwise

Given a theory A, a model M is **A-definable**, it each p_i in M is definable by a formula ψ_i in A

In the context of program verification

- a program satisfies a property iff corresponding CHCs are satisfiable
- verification certificates correspond to models
- counterexamples correspond to derivations of false

IC3, PDR, and Friends (1)

IC3: A SAT-based Hardware Model Checker

- Incremental Construction of Inductive Clauses for Indubitable Correctness
- A. Bradley: SAT-Based Model Checking without Unrolling. VMCAI 2011

PDR: Explained and extended the implementation

- Property Directed Reachability
- N. Eén, A. Mishchenko, R. K. Brayton: Efficient implementation of property directed reachability. FMCAD 2011

PDR with Predicate Abstraction (easy extension of IC3/PDR to SMT)

- A. Cimatti, A. Griggio, S. Mover, St. Tonetta: IC3 Modulo Theories via Implicit Predicate Abstraction. TACAS 2014
- J. Birgmeier, A. Bradley, G. Weissenbacher: Counterexample to Induction-Guided Abstraction-Refinement (CTIGAR). CAV 2014

IC3, PDR, and Friends (2)

GPDR: Non-Linear CHC with Arithmetic constraints

- Generalized Property Directed Reachability
- K. Hoder and N. Bjørner: Generalized Property Directed Reachability. SAT 2012

SPACER: Non-Linear CHC with Arithmetic

- fixes an incompleteness issue in GPDR and extends it with under-approximate summaries
- A. Komuravelli, A. Gurfinkel, S. Chaki: SMT-Based Model Checking for Recursive Programs. CAV 2014

PolyPDR: Convex models for Linear CHC

- simulating Numeric Abstract Interpretation with PDR
- N. Bjørner and A. Gurfinkel: Property Directed Polyhedral Abstraction. VMCAI 2015

ArrayPDR: CHC with constraints over Airthmetic + Arrays

- Required to model heap manipulating programs
- A. Komuravelli, N. Bjørner, A. Gurfinkel, K. L. McMillan:Compositional Verification of Procedural Programs using Horn Clauses over Integers and Arrays. FMCAD 2015

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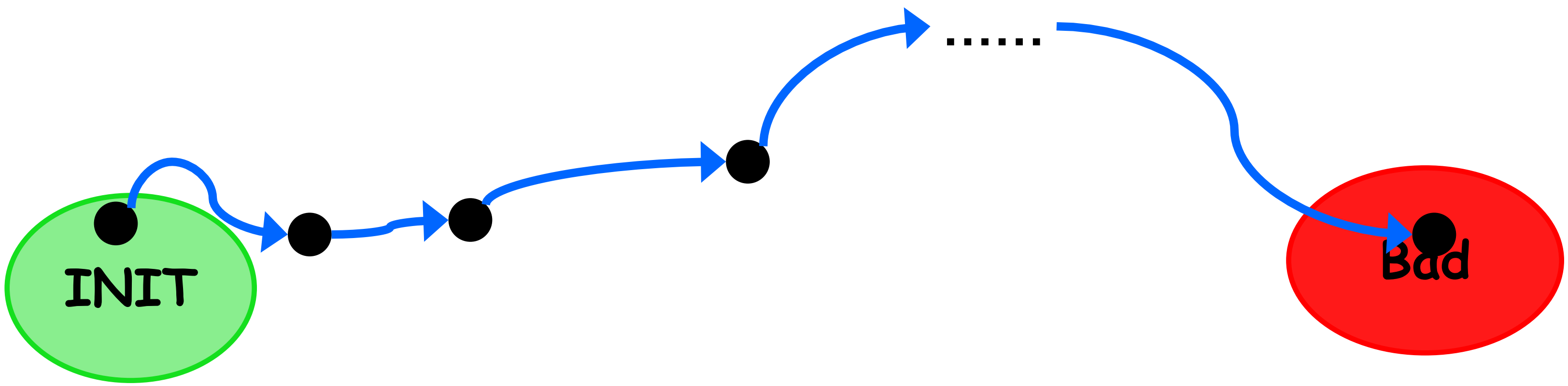
Safety Verification Problem

Is Bad reachable?

Safety Verification Problem

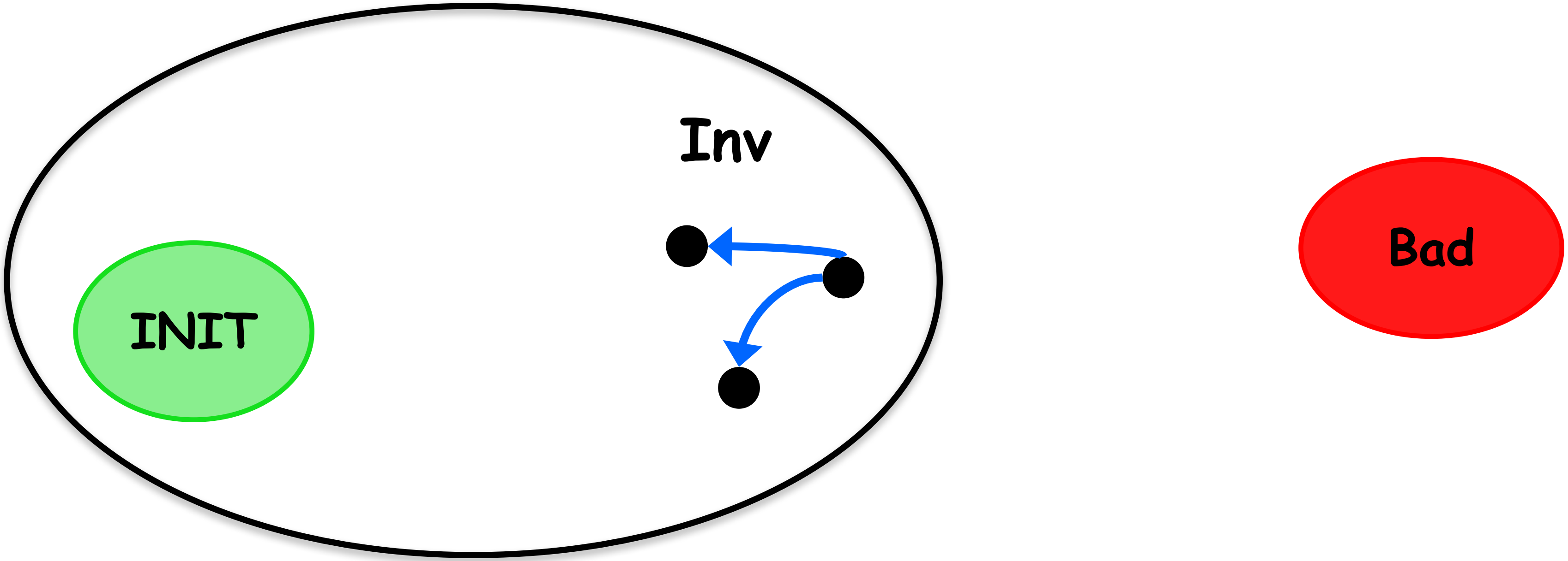
Is Bad reachable?



Yes. There is a counterexample!

Safety Verification Problem

Is Bad reachable?



No. There is an inductive invariant

Programs, Cexs, Invariants

A program *P = (V, Init, $\mathcal{Tr}$, Bad)*

- Notation: $\mathcal{F}$(X) = ∃ ***u*** . (X ∧ $\mathcal{Tr}$) ∨ Init

P is UNSAFE if and only if there exists a number *N* s.t.

$$Init(X_0) \wedge \left(\bigwedge_{i=0}^{N-1} Tr(X_i, X_{i+1}) \right) \wedge Bad(X_N) \;\not\Rightarrow\; \bot$$

P is SAFE if and only if there exists a *safe inductive invariant Inv s.t.*

$$\left.\begin{array}{r} Init \Rightarrow Inv \\ Inv(X) \wedge Tr(X, X') \Rightarrow Inv(X') \end{array}\right\} \quad \text{Inductive}$$

$$Inv \Rightarrow \neg Bad \quad \text{Safe}$$

IC3/PDR Overview

bounded safety

Input: Safety problem $\langle Init(X), Tr(X, X'), Bad(X) \rangle$

$F_0 \leftarrow Init\ ;\ N \leftarrow 0$ **repeat**

$\quad \mathbf{G} \leftarrow \textsc{PdrMkSafe}([F_0, \ldots, F_N], Bad)$

$\quad$ **if** $\mathbf{G} = [\,]$ **then return** *Reachable*;

$\quad \forall 0 \leq i \leq N \cdot F_i \leftarrow \mathbf{G}[i]$

$\quad F_0, \ldots, F_N \leftarrow \textsc{PdrPush}([F_0, \ldots, F_N])$

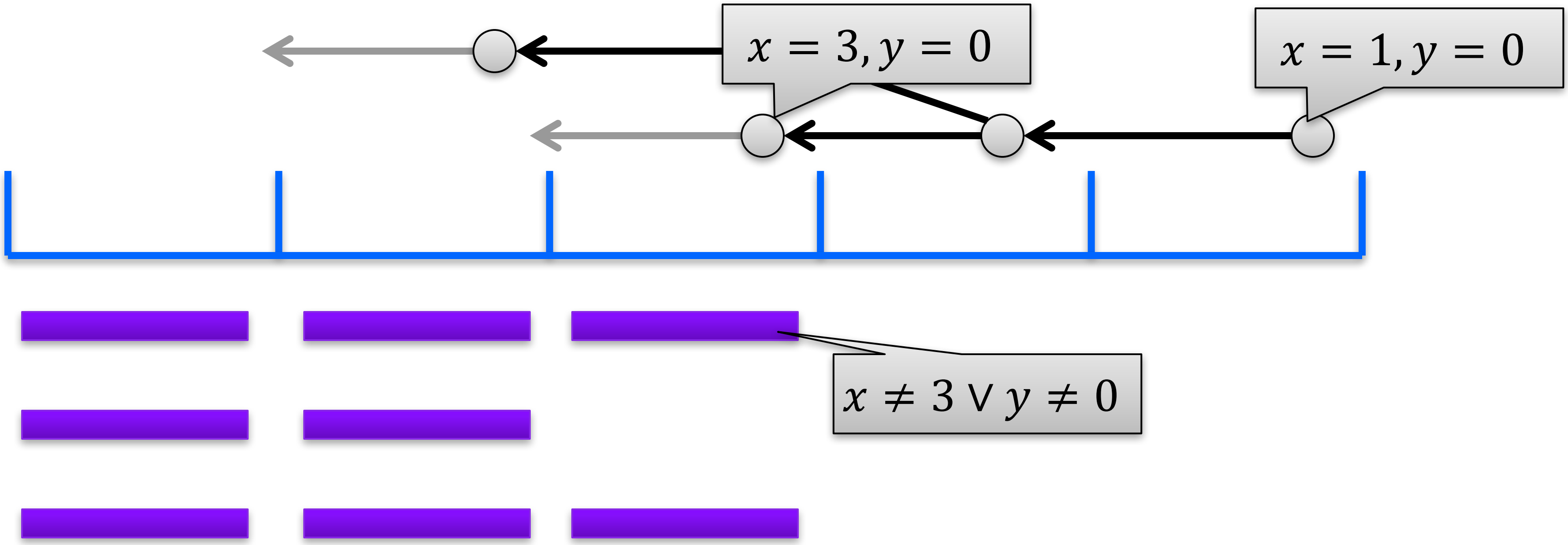
$\quad$ **if** $\exists 0 \leq i < N \cdot F_i = F_{i+1}$ **then return** *Unreachable*;

$\quad N \leftarrow N + 1\ ;\ F_N \leftarrow \emptyset$

until ∞;

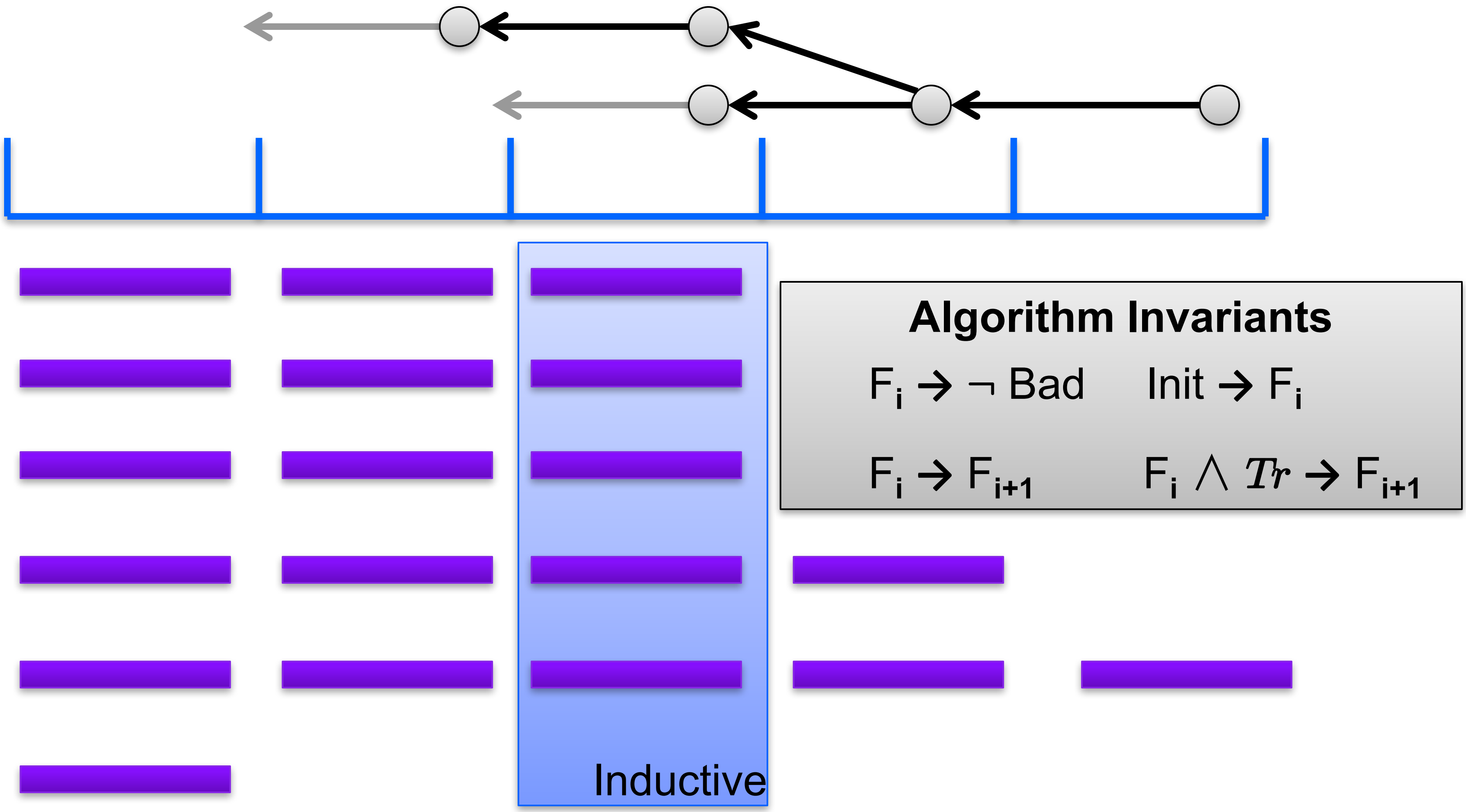
strengthen result

IC3/PDR In Pictures: MkSafe



Push

IC3/PDR in Pictures: Push



IC3/PDR: Solving Linear (Propositional) CHC

Unreachable and Reachable

- terminate the algorithm when a solution is found

Unfold

- increase search bound by 1

Candidate

- choose a bad state in the last frame

Decide

- extend a cex (backward) consistent with the current frame
- choose an assignment ***s*** s.t. $(s \wedge R_i \wedge Tr \wedge cex')$ is SAT

Conflict

- construct a lemma to explain why cex cannot be extended
- Find a clause ***L*** s.t. $L \Rightarrow \neg cex$, $Init \Rightarrow L$, and $L \wedge R_i \wedge Tr \Rightarrow L'$

Induction

- propagate a lemma as far into the future as possible
- (optionally) strengthen by dropping literals

Decide Rule: Generalizing Predecessors

Decide If $\langle m, i+1\rangle \in Q$ and there are m_0 and m_1 s.t. $m_1 \to m$, $m_0 \wedge m_1'$ is satisfiable, and $m_0 \wedge m_1' \to F_i \wedge Tr \wedge m'$, then add $\langle m_0, i\rangle$ to Q.

Decide rule chooses a (generalized) predecessor m_0 of m that is consistent with the current frame

Simplest implementation is to extract a predecessor m_o from a satisfying assignment of $M \models F_i \wedge Tr \wedge m'$

- m_0 cab be further generalized using ternary simulation by dropping literals and checking that m' remains forced

An alternative is to let m_0 be an implicant (not necessarily prime) of $F_i \wedge \exists X'.(Tr \wedge m')$

- finding a prime implicant is difficult because of the existential quantification
- we settle for an arbitrary implicant. The side conditions ensure it is not trivial

Conflict Rule: Inductive Generalization

Conflict For $0 \leq i < N$: given a candidate model $\langle m, i+1 \rangle \in Q$ and clause φ, such that $\varphi \to \neg m$, if $Init \to \varphi$, and $\varphi \wedge F_i \wedge Tr \to \varphi'$, then add φ to F_j, for $j \leq i+1$.

A clause φ is inductive relative to F iff

- Init → φ (Initialization) and φ ∧ F ∧ Tr → φ (Inductiveness)

Implemented by first letting φ = ¬m and generalizing φ by iteratively dropping literals while checking the inductiveness condition

Theorem: Let F_0, F_1, …, F_N be a valid IC3 trace. If φ is inductive relative to F_i, $0 \cdot i < N$, then, for all $j \cdot i$, φ is inductive relative to F_j.

- Follows from the monotonicity of the trace
  - if j < i then $F_j \to F_i$
  - if $F_j \to F_i$ then $(\varphi \wedge F_i \wedge Tr \to \varphi) \to (\varphi \wedge F_j \wedge Tr \to \varphi')$

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From Propositional PDR to Solving CHC

Infinite Theories

- infinitely many satisfying assignments
- can’t simply enumerate (in decide)
- can’t block one assignment at a time (in conflict)

Non-Linear Horn Clauses

- multiple predecessors (in decide)

The problem is undecidable in general, but we want an algorithm that makes progress

- don’t get stuck in a decidable fragment

PDR FOR ARITHMETIC CHC

IC3/PDR: Solving Linear (Propositional) CHC

Unreachable and Reachable

- terminate the algorithm when a solution is found

Unfold

- increase search bound by 1

Candidate

- choose a bad state in the last frame

Theory dependent

Decide

- extend a cex (backward) consistent with the current frame
- choose an assignment ***s*** s.t. $(s \wedge R_i \wedge Tr \wedge cex')$ is SAT

Conflict

- construct a lemma to explain why cex cannot be extended
- Find a clause ***L*** s.t. $L \Rightarrow \neg cex$, $Init \Rightarrow L$, and $L \wedge R_i \wedge Tr \Rightarrow L'$

Induction

- propagate a lemma as far into the future as possible
- (optionally) strengthen by dropping literals

$$((F_i \wedge Tr) \vee Init') \Rightarrow \varphi'$$

$$\varphi' \Rightarrow \neg c'$$

Looking for φ’

ARITHMETIC CONFLICT

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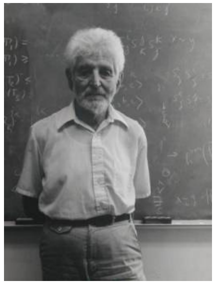
Craig Interpolation Theorem



Theorem (Craig 1957)
Let A and B be two First Order (FO) formulae such that $A \Rightarrow \neg B$, then there exists a FO formula I, denoted ITP(A, B), such that

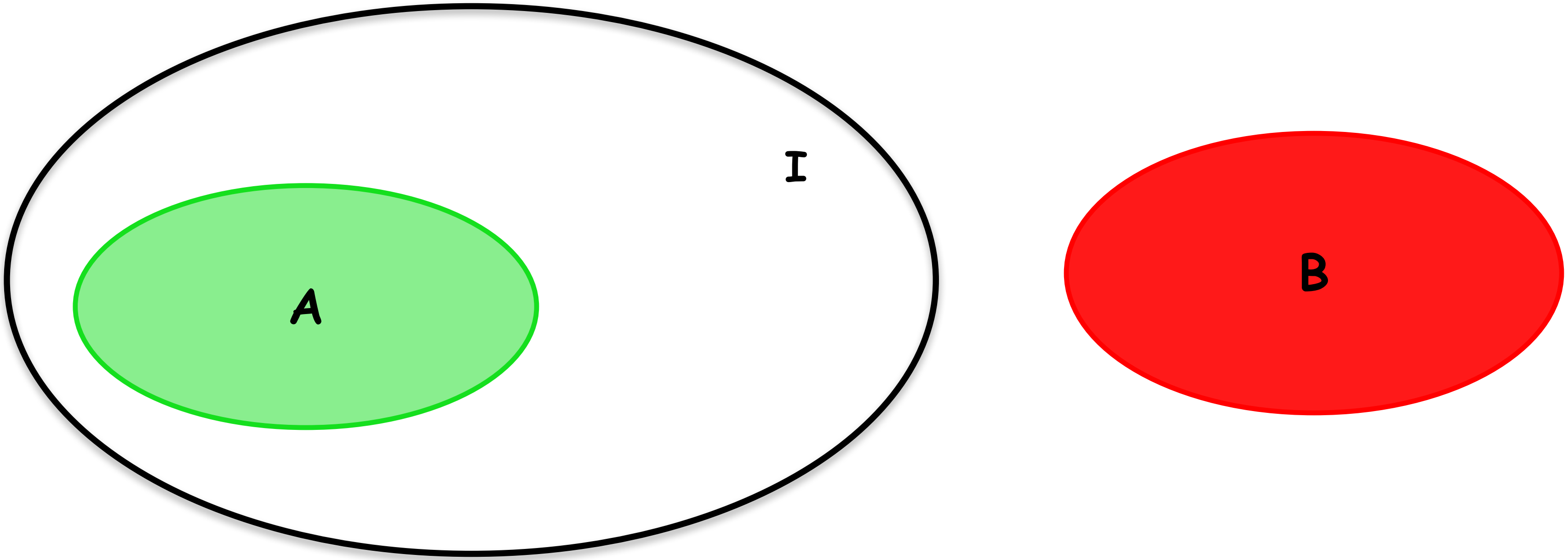
$$A \Rightarrow I \qquad I \Rightarrow \neg B$$

$$\mathit{atoms}(I) \in \mathit{atoms}(A) \cap \mathit{atoms}(B)$$

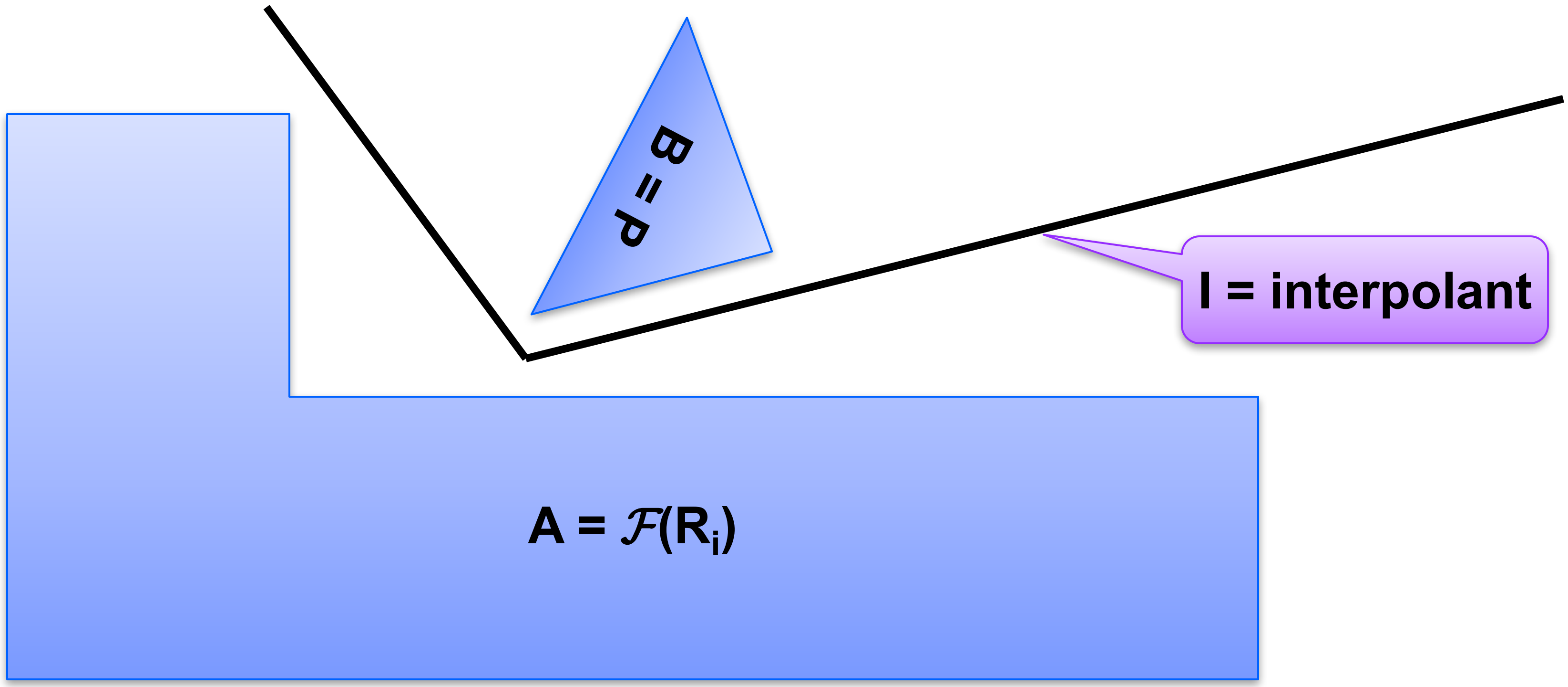
A Craig interpolant ITP(A, B) can be effectively constructed from a resolution proof of unsatisfiability of $A \wedge B$

In Model Checking, Craig Interpolation Theorem is used to safely over-approximate the set of (finitely) reachable states

Craig Interpolant



Craig Interpolation for Linear Arithmetic



Useful properties of existing interpolation algorithms [CGS10] [HB12]

- I $\in$ ITP (A, B) then $\neg$I $\in$ ITP (B, A)
- if A is syntactically convex (a monomial), then I is convex
- if B is syntactically convex, then I is co-convex (a clause)
- if A and B are syntactically convex, then I is a half-space

Arithmetic Conflict

Notation: $\mathcal{F}(A) = (A(X) \wedge \mathit{Tr}) \vee \mathit{Init}(X')$.

Conflict For $0 \leq i < N$, given a counterexample $\langle P, i+1 \rangle \in Q$ s.t. $\mathcal{F}(F_i) \wedge P'$ is unsatisfiable, add $P^{\uparrow} = \textsc{Itp}(\mathcal{F}(F_i), P')$ to F_j for $j \leq i+1$.

Counterexample is blocked using Craig Interpolation

- summarizes the reason why the counterexample cannot be extended

Generalization is not inductive

- weaker than IC3/PDR
- inductive generalization for arithmetic is still an open problem

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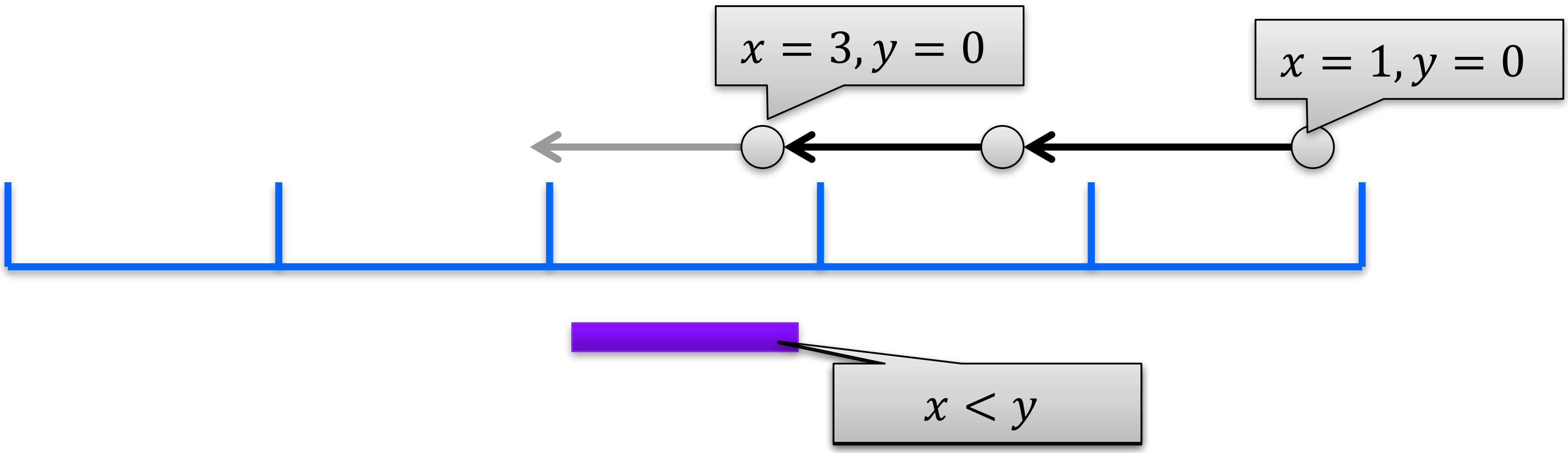
IC3/PDR In Pictures: MkSafe

MkSafe



Computing Interpolants for IC3/PDR

Much simpler than general interpolation problem for $A \wedge B$

- B is always a conjunction of literals
- A is dynamically split into DNF by the SMT solver
- DPLL(T) proofs do not introduce new literals

Interpolation algorithm is reduced to analyzing all theory lemmas in a DPLL(T) proof produced by the solver

- every theory-lemma that mixes B-pure literals with other literals is interpolated to produce a single literal in the final solution
- interpolation is restricted to clauses of the form $(\bigwedge B_i \Rightarrow \vee A_j)$

Interpolating (UNSAT) Cores *(ongoing work with Bernhard Gleiss)*

- improve interpolation algorithms and definitions to the specific case of PDR
- classical interpolation focuses on eliminating non-shared literals
- in PDR, the focus is on finding good generalizations

$$s \subseteq pre(c)$$

$$\equiv \quad s \Rightarrow \exists X' \,.\, Tr \wedge c'$$

Computing a predecessor ***s*** of a counterexample ***c***

ARITHMETIC DECIDE

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Model Based Projection

Definition: Let φ be a formula, U a set of variables, and M a model of φ. Then ψ = MBP (U, M, φ) is a Model Based Projection of U, M and φ iff

1. ψ is a monomial
2. Vars(ψ) $\subseteq$ Vars(φ) \ U
3. M ⊧ ψ
4. $\psi \Rightarrow \exists$ U . φ

Model Based Projection under-approximates existential quantifier elimination relative to a given model (i.e., satisfying assignment)

Loos-Weispfenning Quantifier Elimination

φ is LRA formula in Negation Normal Form

E is set of x=t atoms, U set of x < t atoms, and L set of s < x atoms

There are no other occurrences of x in φ[x]

$$\exists x . \varphi[x] \equiv \varphi[\infty] \lor \bigvee_{x=t \in E} \varphi[t] \lor \bigvee_{x<t \in U} \varphi[t - \epsilon]$$

where

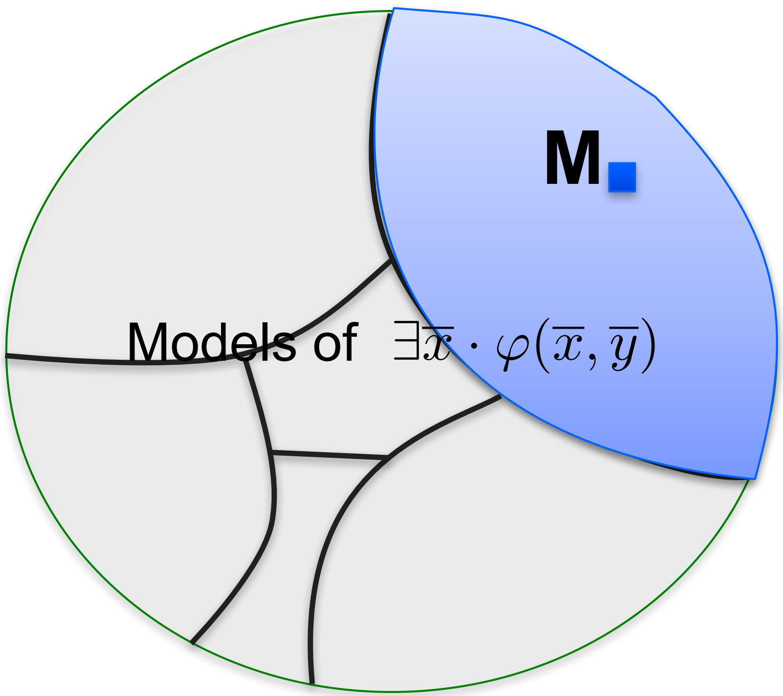
$$(x < t')[t - \epsilon] \equiv t \leq t' \qquad (s < x)[t - \epsilon] \equiv s < t \qquad (x = e)[t - \epsilon] \equiv \mathit{false}$$

The case of lower bounds is dual

- using –∞ and t+ϵ

Model Based Projection

Expensive to find a quantifier-free $\psi(\overline{y}) \equiv \exists \overline{x} \cdot \varphi(\overline{x}, \overline{y})$



1. Find model M of φ (x,y)

2. Compute a partition containing M

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MBP for Linear Rational Arithmetic

Compute a **single** disjunct from LW-QE that includes the model

- Use the Model to uniquely pick a substitution term for x

$$Mbp_x(M, x = s \wedge L) = L[x \leftarrow s]$$

$$Mbp_x(M, x \neq s \wedge L) = Mbp_x(M, s < x \wedge L) \text{ if } M(x) > M(s)$$

$$Mbp_x(M, x \neq s \wedge L) = Mbp_x(M, -s < -x \wedge L) \text{ if } M(x) < M(s)$$

$$Mbp_x(M, \bigwedge_i s_i < x \wedge \bigwedge_j x < t_j) = \bigwedge_i s_i < t_0 \wedge \bigwedge_j t_0 \leq t_j \text{ where } M(t_0) \leq M(t_i), \forall i$$

MBP techniques have been developed for

- Linear Rational Arithmetic, Linear Integer Arithmetic
- Theories of Arrays, and Recursive Data Types

Arithmetic Decide

Notation: $\mathcal{F}(A) = (A(X) \wedge Tr(X, X') \vee Init(X')$.

Decide If $\langle P, i+1\rangle \in Q$ and there is a model $m(X, X')$ s.t. $m \models \mathcal{F}(F_i) \wedge P'$, add $\langle P_{\downarrow}, i\rangle$ to Q, where $P_{\downarrow} = \mathrm{MBP}(X', m, \mathcal{F}(F_i) \wedge P')$.

Compute a predecessor using an under-approximation of quantifier elimination – called Model Based Projection

To ensure progress, Decide must be finite

- finitely many possible predecessors when all other arguments are fixed

Alternatives

- Completeness can follow from the **Conflict** rule only
  - for Linear Arithmetic this means using Fourier-Motzkin implicants
- Completeness can follow from an interaction of **Decide** and **Conflict**

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PDR FOR NON-LINEAR CHC

Non-Linear CHC Satisfiability

Satisfiability of a set of arbitrary (i.e., linear or non-linear) CHCs is reducible to satisfiability of THREE clauses of the form

$$Init(X) \to P(X)$$

$$P(X) \to !Bad(X)$$

$$P(X) \wedge P(X^o) \wedge Tr(X, X^o, X') \to P(X')$$

where, $X' = \{x' \mid x \in X\}$, $X^o = \{x^o \mid x \in X\}$, P a fresh predicate, and Init, Bad, and Tr are constraints

Generalized GPDR

Input: A safety problem $\langle Init(X), Tr(X, X^o, X'), Bad(X) \rangle$.
Output: *Unreachable* or *Reachable*
Data: A cex queue Q, where a cex $\langle c_0, \ldots, c_k \rangle \in Q$ is a tuple, each $c_j = \langle m, i \rangle$, m is a cube over state variables, and $i \in \mathbb{N}$. A level N. A trace $F_0, F_1, \ldots$
Notation: $\mathcal{F}(A, B) = Init(X') \vee (A(X) \wedge B(X^o) \wedge Tr)$, and $\mathcal{F}(A) = \mathcal{F}(A, A)$
Initially: $Q = \emptyset$, $N = 0$, $F_0 = Init$, $\forall i > 0 \cdot F_i = \emptyset$
Require: $Init \to \neg Bad$
repeat

- **Unreachable** If there is an $i < N$ s.t. $F_i \subseteq F_{i+1}$ **return** *Unreachable*.
- **Reachable** if exists $t \in Q$ s.t. for all $\langle c, i \rangle \in t$, $i = 0$, **return** *Reachable*.
- **Unfold** If $F_N \to \neg Bad$, then set $N \leftarrow N + 1$ and $Q \leftarrow \emptyset$.
- **Candidate** If for some m, $m \to F_N \wedge Bad$, then add $\langle \langle m, N \rangle \rangle$ to Q.
- **Decide** If there is a $t \in Q$, with $c = \langle m, i+1 \rangle \in t$, $m_1 \to m$, $l_0 \wedge m_0^o \wedge m_1'$ is satisfiable, and $l_0 \wedge m_0^o \wedge m_1' \to F_i \wedge F_i^o \wedge Tr \wedge m'$ then add $\hat{t}$ to Q, where $\hat{t} = t$ with c replaced by two tuples $\langle l_0, i \rangle$, and $\langle m_0, i \rangle$.
- **Conflict** If there is a $t \in Q$ with $c = \langle m, i+1 \rangle \in t$, s.t. $\mathcal{F}(F_i) \wedge m'$ is unsatisfiable. Then, add $\varphi = \textsc{Itp}(\mathcal{F}(F_i), m')$ to F_j, for all $0 \le j \le i + 1$.
- **Leaf** If there is $t \in Q$ with $c = \langle m, i \rangle \in t$, $0 < i < N$ and $\mathcal{F}(F_{i-1}) \wedge m'$ is unsatisfiable, then add $\hat{t}$ to Q, where $\hat{t}$ is t with c replaced by $\langle m, i+1 \rangle$.
- **Induction** For $0 \le i < N$ and a clause $(\varphi \vee \psi) \in F_i$, if $\varphi \notin F_{i+1}$, $\mathcal{F}(\phi \wedge F_i) \to \phi'$, then add φ to F_j, for all $j \le i + 1$.

until ∞;

counterexample is a tree

two predecessors

theory-aware **Conflict**

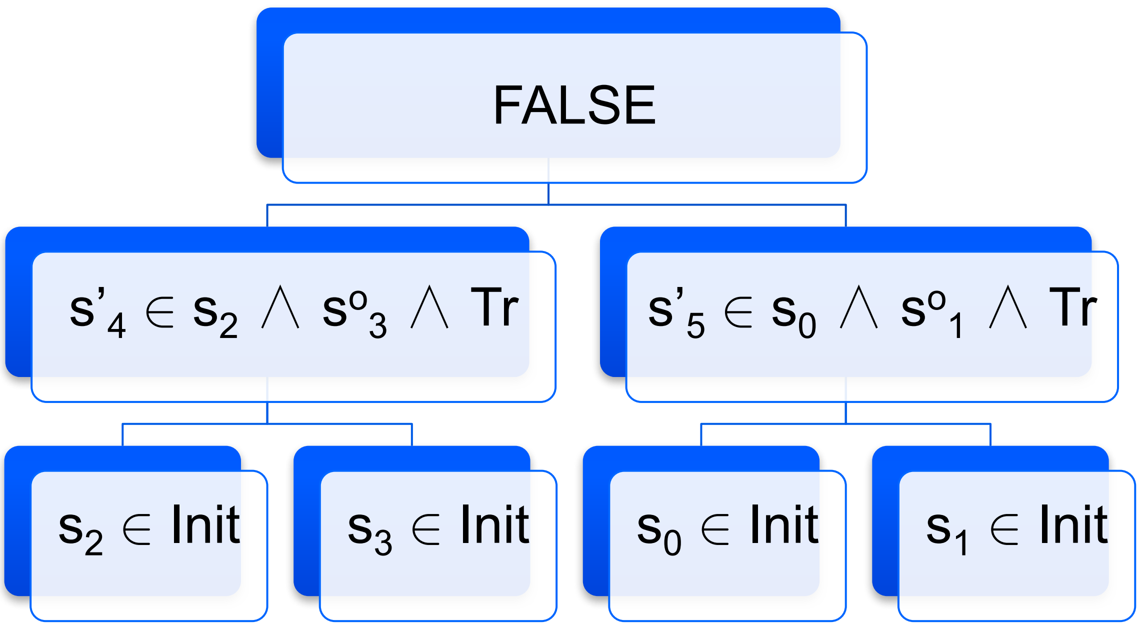
Counterexamples to non-linear CHC

A set S of CHC is unsatisfiable iff S can derive FALSE

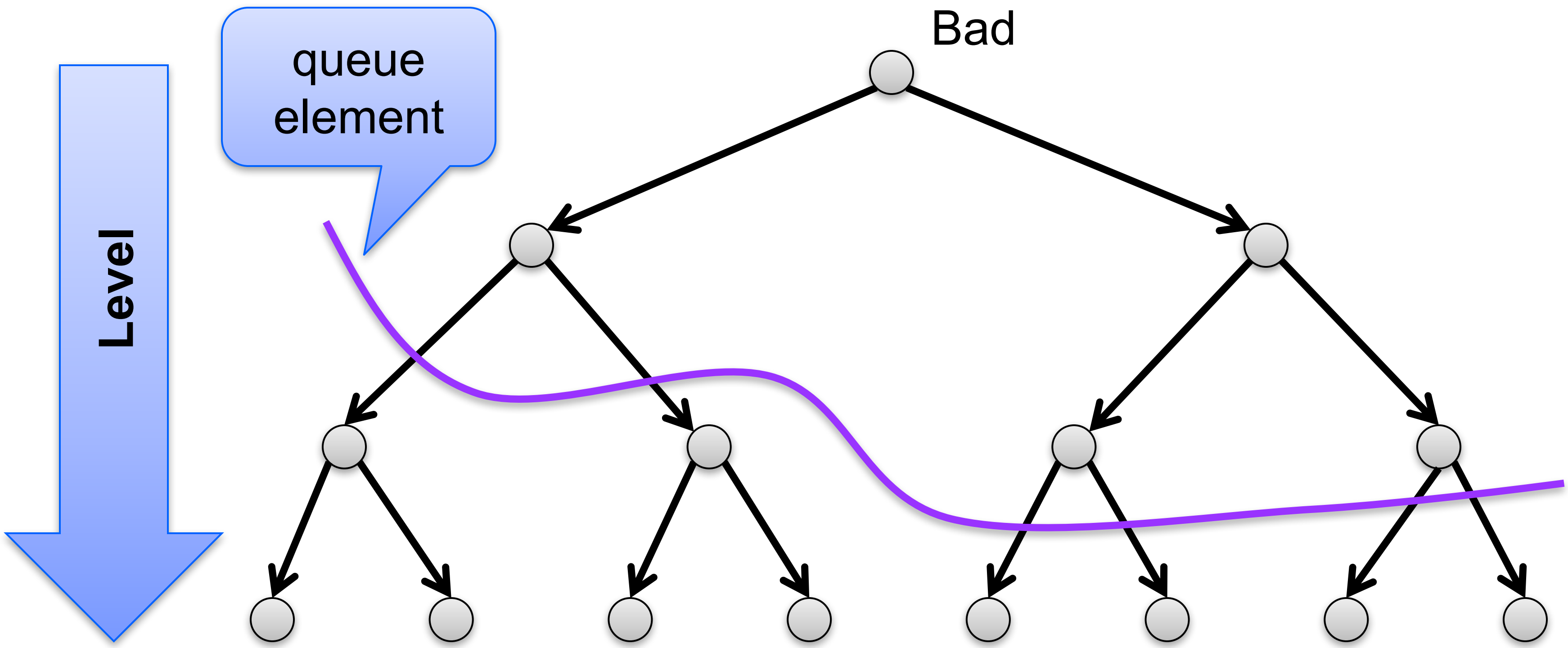
- we call such a derivation a counterexample

For linear CHC, the counterexample is a path

For non-linear CHC, the counterexample is a tree



GPDR Search Space



At each step, one CTI in the frontier is chosen and its two children are expanded

GPDR: Deciding predecessors

Decide If there is a $t \in Q$, with $c = \langle m, i+1 \rangle \in t$, $m_1 \rightarrow m$, $l_0 \wedge m_0^o \wedge m_1'$ is satisfiable, and $l_0 \wedge m_0^o \wedge m_1' \rightarrow F_i \wedge F_i^o \wedge Tr \wedge m'$ then add $\hat{t}$ to Q, where $\hat{t} = t$ with c replaced by two tuples $\langle l_0, i \rangle$, and $\langle m_0, i \rangle$.

Compute two predecessors at each application of **GPDR/Decide**

Can explore both predecessors in parallel

- e.g., BFS or DFS exploration order

Number of predecessors is unbounded

- incomplete even for finite problem (i.e., non-recursive CHC)

No caching/summarization of previous decisions

- worst-case exponential for Boolean Push-Down Systems

Spacer

Same queue as in IC3/PDR

Cache Reachable states

Three variants of **Decide**

Same **Conflict** as in APDR/GPDR

Input: A safety problem $\langle Init(X), Tr(X, X^o, X'), Bad(X)\rangle$.
Output: *Unreachable* or *Reachable*
Data: A cex queue Q, where a cex $c \in Q$ is a pair $\langle m, i\rangle$, m is a cube over state variables, and $i \in \mathbb{N}$. A level N. A set of reachable states $\textsc{Reach}$. A trace $F_0, F_1, \ldots$
Notation: $\mathcal{F}(A, B) = Init(X') \vee (A(X) \wedge B(X^o) \wedge Tr)$, and $\mathcal{F}(A) = \mathcal{F}(A, A)$
Initially: $Q = \emptyset$, $N = 0$, $F_0 = Init$, $\forall i > 0 \cdot F_i = \emptyset$, $\textsc{Reach} = Init$
Require: $Init \rightarrow \neg Bad$
repeat

Unreachable If there is an $i < N$ s.t. $F_i \subseteq F_{i+1}$ **return** *Unreachable*.

Reachable If $\textsc{Reach} \wedge Bad$ is satisfiable, **return** *Reachable*.

Unfold If $F_N \rightarrow \neg Bad$, then set $N \leftarrow N + 1$ and $Q \leftarrow \emptyset$.

Candidate If for some m, $m \rightarrow F_N \wedge Bad$, then add $\langle m, N\rangle$ to Q.

Successor If there is $\langle m, i+1\rangle \in Q$ and a model M $M \models \psi$, where $\psi = \mathcal{F}(\vee\textsc{Reach}) \wedge m'$. Then, add s to $\textsc{Reach}$, where $s' \in \text{MBP}(\{X, X^o\}, \psi)$.

DecideMust If there is $\langle m, i+1\rangle \in Q$, and a model M $M \models \psi$, where $\psi = \mathcal{F}(F_i, \vee\textsc{Reach}) \wedge m'$. Then, add s to Q, where $s \in \text{MBP}(\{X^o, X'\}, \psi)$.

DecideMay If there is $\langle m, i+1\rangle \in Q$ and a model M $M \models \psi$, where $\psi = \mathcal{F}(F_i) \wedge m'$. Then, add s to Q, where $s^o \in \text{MBP}(\{X, X'\}, \psi)$.

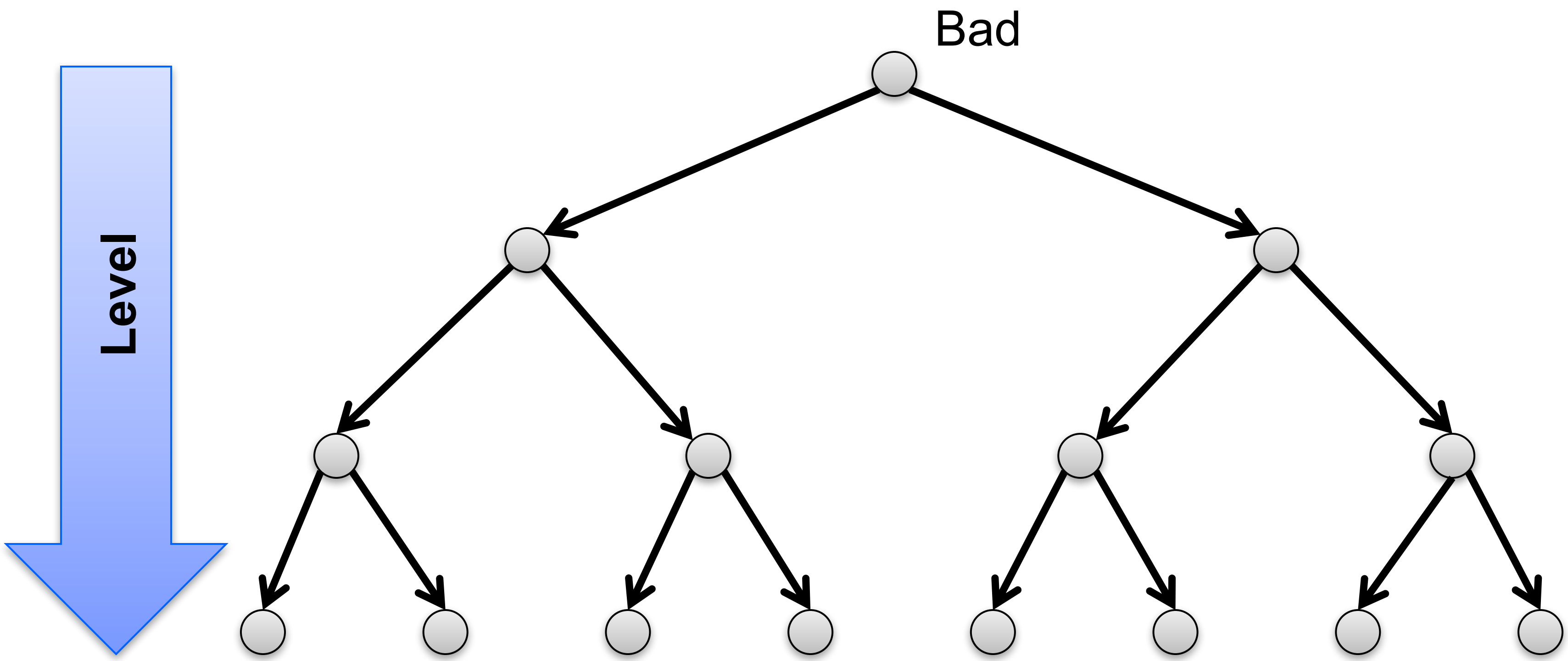
Conflict If there is an $\langle m, i+1\rangle \in Q$, s.t. $\mathcal{F}(F_i) \wedge m'$ is unsatisfiable. Then, add $\varphi = \textsc{Itp}(\mathcal{F}(F_i), m')$ to F_j, for all $0 \le j \le i+1$.

Leaf If $\langle m, i\rangle \in Q$, $0 < i < N$ and $\mathcal{F}(F_{i-1}) \wedge m'$ is unsatisfiable, then add $\langle m, i+1\rangle$ to Q.

Induction For $0 \le i < N$ and a clause $(\varphi \vee \psi) \in F_i$, if $\varphi \notin F_{i+1}$, $\mathcal{F}(\phi \wedge F_i) \rightarrow \phi'$, then add φ to F_j, for all $j \le i+1$.

until ∞;

SPACER Search Space



Unfold the derivation tree in a fixed depth-first order

- use MBP to decide on counterexamples

Learn new facts (reachable states) on the way up

- use MBP to propagate facts bottom up

Successor Rule: Computing Reachable States

Successor If there is $\langle m, i+1 \rangle \in Q$ and a model M $M \models \psi$, where $\psi = \mathcal{F}(\vee\textsc{Reach}) \wedge m'$. Then, add s to $\textsc{Reach}$, where $s' \in \mathrm{MBP}(\{X, X^o\}, \psi)$.

Computing new reachable states by under-approximating forward image using MBP

- since MBP is finite, guarantee to exhaust all reachable states

Second use of MBP

- orthogonal to the use of MBP in Decide
- REACH can contain auxiliary variables, but might get too large

For Boolean CHC, the number of reachable states is bounded

- complexity is polynomial in the number of states
- same as reachability in Push Down Systems

Decide Rule: Must and May refinement

DecideMust If there is $\langle m, i+1 \rangle \in Q$, and a model M $M \models \psi$, where $\psi = \mathcal{F}(F_i, \vee\textsc{Reach}) \wedge m'$. Then, add s to Q, where $s \in \text{MBP}(\{X^o, X'\}, \psi)$.

DecideMay If there is $\langle m, i+1 \rangle \in Q$ and a model M $M \models \psi$, where $\psi = \mathcal{F}(F_i) \wedge m'$. Then, add s to Q, where $s^o \in \text{MBP}(\{X, X'\}, \psi)$.

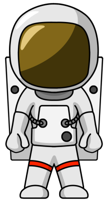
DecideMust

- use computed summary to skip over a call site

DecideMay

- use over-approximation of a calling context to guess an approximation of the call-site
- the call-site either refutes the approximation (**Conflict**) or refines it with a witness (**Successor**)

Conclusion and Future Work



Spacer: an SMT-based procedure for deciding CHC modulo theories

- extends IC3/PDR from SAT to SMT
- interpolation to over-approximate a possible model
- model-based projection to summarize derivations

The curse of interpolation

- interpolation is fantastic at quickly discovering good lemmas
- BUT it is highly unstable: small changes to input (or code) drastically change what is discovered
- what is easy today might be difficult tomorrow ☹

Harnessing the power of parallelism (see FMCAD'17)

- Spacer is highly non-deterministic: many sound choices for bounded exploration and lemma generation
- Lemmas (invariants) are easy to share between multiple instances
- Problems are naturally partitioned in Decide rule

Farkas Lemma

Let $M = t_1 \geq b_1 \wedge \ldots \wedge t_n \geq b_n$, where t_i are linear terms and b_i are constants M is *unsatisfiable* iff $0 \geq 1$ is derivable from M by resolution

M is *unsatisfiable* iff $M \vdash 0 \geq 1$

- e.g., $x + y > 10, -x > 5, -y > 3 \vdash (x+y-x-y) > (10 + 5 + 3) \vdash 0 > 18$

M is unsatisfiable iff there exist *Farkas* coefficients $g_1, \ldots, g_n$ such that

- $g_i \geq 0$
- $g_1 \times t_1 + \ldots + g_n \times t_n = 0$
- $g_1 \times b_1 + \ldots + g_n \times b_n \geq 1$

Interpolation for Linear Real Arithmetic

Let $M = A \wedge B$ be UNSAT, where

- $A = t_1 \geq b_1 \wedge \ldots \wedge t_i \geq b_i$, and
- $B = t_{i+1} \geq b_i \wedge \ldots \wedge t_n \geq b_n$

Let $g_1, \ldots, g_n$ be the Farkas coefficients witnessing UNSAT

Then

- $g_1 \times (t_1 \geq b_1) + \ldots + g_i \times (t_i \geq b_i)$ is an interpolant between A and B
- $g_{i+1} \times (t_{i+1} \geq b_i) + \ldots + g_n \times (t_n \geq b_n)$ is an interpolant between B and A

- $g_1 \times t_1 + \ldots + g_i \times t_i = -(g_{i+1} \times t_{i+1} + \ldots + g_n \times t_n)$
- $\neg(g_{i+1} \times (t_{i+1} \geq b_i) + \ldots + g_n \times (t_n \geq b_n))$ is an interpolant between A and B

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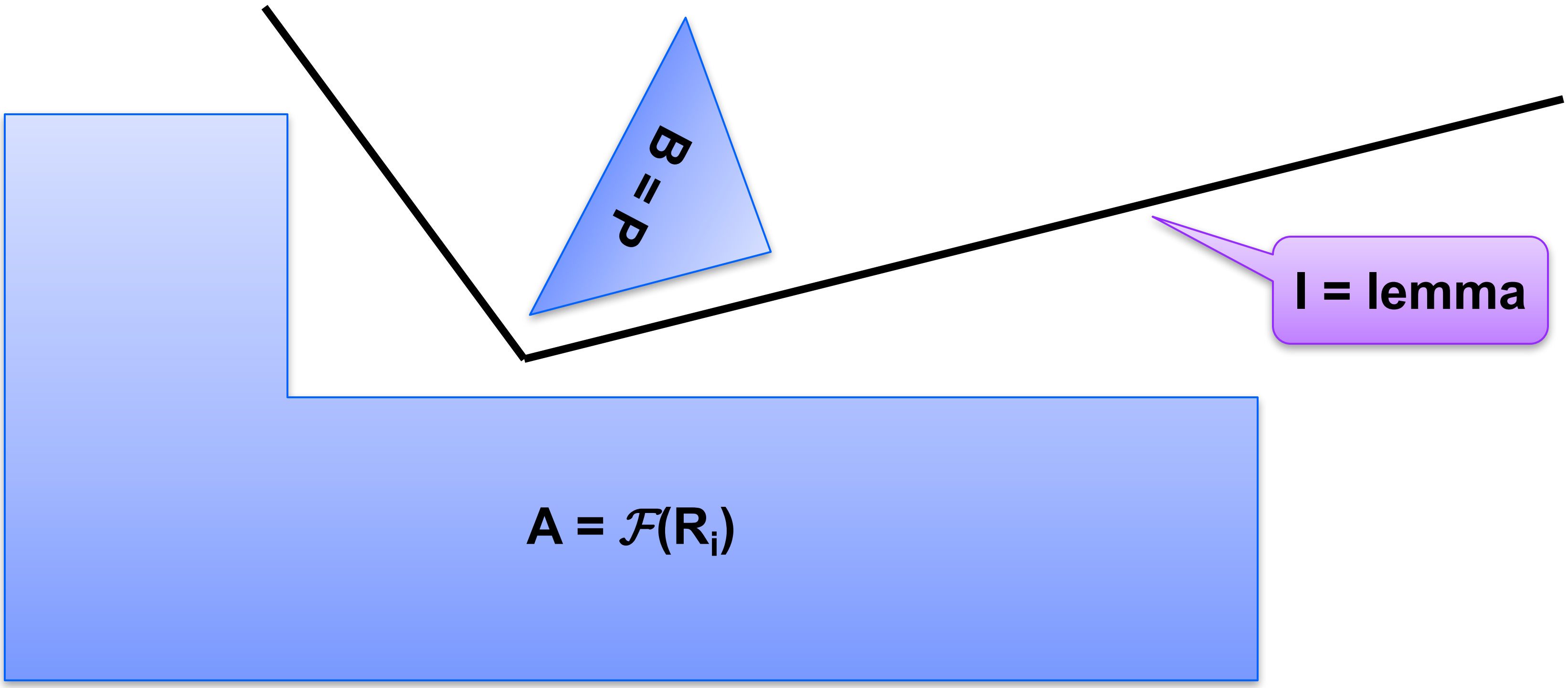
Craig Interpolation for Linear Arithmetic



Useful properties of existing interpolation algorithms [CGS10] [HB12]

- $I \in ITP(A, B)$ then $\neg I \in ITP(B, A)$
- if A is syntactically convex (a monomial), then I is convex
- if B is syntactically convex, then I is co-convex (a clause)
- if A and B are syntactically convex, then I is a half-space