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THE COVERING AND BOUNDEDNESS PROBLEMS FOR VECTOR ADDITION SYSTEMS

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Abstract. New decision procedures for the covering and boundedness problems for vector addition systems are obtained. These procedures require at most space $2^{cn \log n}$ for some constant c. The procedures nearly achieve recently established lower bounds on the amount of space inherently required to solve these problems, and so are much more efficient than previously known non-primitive-recursive decision procedures.

1. Introduction

Petri nets and vector addition systems have been studied as models of asynchronous processes, with attempts to understand the mathematical properties of these systems. Efficient decision procedures are presented here for the covering problem and for the boundedness problem for vector addition systems.

These procedures operate in space exponential in the size of the input. By some lower bound results of Lipton [2, 5], the space complexity of these procedures is nearly optimal. Karp and Miller [4] have previously shown that both of these problems are decidable, but their procedures do not operate in even primitive recursive space.

2. Notation and definitions

We let $\mathbf{Z}$ represent the integers, $\mathbf{N}$ the nonnegative integers, and $\mathbf{N}^+$ the positive integers. If $v \in \mathbf{Z}^k$ $(k \in \mathbf{N}^+)$ is a vector, then by $v(i)$ we mean the i^{th} place of v, for $1 \leqslant i \leqslant k$. If $v_1, v_2 \in \mathbf{Z}^k$, then we define $v_1 + v_2$ to be that vector $v \in \mathbf{Z}^k$ such that $v(i) = v_1(i) + v_2(i)$ for all i, $1 \leqslant i \leqslant k$; $v_1 - v_2$ is defined similarly. We say that $v_1 \leqslant v_2$ if $v_1(i) \leqslant v_2(i)$ for all i, $1 \leqslant i \leqslant k$. If $v_1 \leqslant v_2$ but $v_1 \neq v_2$, then we write $v_1 < v_2$. We say that v is nonnegative if $v \in \mathbf{N}^k$. We use $\bar{0}$ to denote the zero-vector, where the dimension will be clear from the context.

Definition 2.1. A *vector addition system* consists of a *dimension* $k \in \mathbf{N}^+$, and a pair (v, A) where $v \in \mathbf{Z}^k$ is the *start vector*, and $A \subseteq \mathbf{Z}^k$ is a finite set called the *addition set*. A finite sequence of vectors $w_1, w_2, \ldots, w_m \in \mathbf{Z}^k$ is said to be a *path in* (v, A) *of length* m if $w_1 = v$ and if $w_{i+1} - w_i \in A$ for all i $1 \leq i \leq m$. If there is a path in (v, A) ending in $w \in \mathbf{N}^k$, with only nonnegative vectors on it, then we say that w is *reachable* in (v, A); $R(v, A) = \{w \in \mathbf{N}^k\colon\ w$ is reachable in $(v, A)\}$ is called the *reachability set* of (v, A).

$C(v, A) = \{w \in \mathbf{N}^k\colon$ for some $v_1 \in R(v, A)$, $w \leq v_1\}$ is called the *coverability set* of (v, A).

Since we will be considering the complexity of algorithms for problems involving vector addition systems, it is necessary to have a precise idea of the size of such a system. We will say that the size of a vector is the sum of the lengths of the binary representations of the components (where the length of 0 is 1). The size of a finite set of vectors is the sum of the sizes of the members.

The covering problem for vector addition systems is to determine for system (v, A) and vector $v_1 \in \mathbf{N}^k$, if $v_1 \in C(v, A)$. In Section 3 we present a decision procedure for this problem which operates in space $2^{cn \log n}$ (for some constant c) as a function of the length of input.

The boundedness problem is to determine for system (v, A) if $R(v, A)$ is finite. The decision procedure presented in Section 4 for this problem also operates in space $2^{cn \log n}$.

Lipton [5] has shown that for some constant $d > 0$, neither of these problems can be decided in space $2^{d\sqrt{n}}$, and hence our procedures are close to being optimal in their use of storage. Lipton's lower bounds are valid even if one only considers input whose vectors have components of value -1, 0, or 1.

Remark 2.2. Actually Lipton shows a lower bound of 2^{dn} for a slightly different formulation of these problems; this translates to a lower bound of $2^{d\sqrt{n}}$ for our formulation.

3. The covering problem

Let A be a fixed addition set of size $\leq n$ and dimension $k \leq n$; let $v_1 \in \mathbf{N}^k$ be a fixed vector of size $\leq n$. We will show that for any $v \in \mathbf{N}^k$, in order to find a $v_2 \in R(v, A)$ such that $v_2 \geq v_1$, it suffices to look at paths in (v, A) of length $\leq 2^{2^{cn \log n}}$.

Definition 3.1. Let $w \in \mathbf{Z}^k$, let $0 \leq i \leq k$. Then we say that w is *i-bounded* if $w(j) \geq 0$ for $1 \leq j \leq i$. If $r \in \mathbf{N}^+$ is such that $0 \leq w(j) < r$ for $1 \leq j \leq i$, then we say that w is *i–r bounded*. Let $p = w_1, w_2, \ldots, w_m$ be a sequence of members of $\mathbf{Z}^k$. We say p is

i-bounded (i--r bounded) if every member of p is i-bounded (i--r bounded). If $w_n(j) \geq v_1(j)$ for $1 \leq j \leq i$, then we say that p is an *i-covering sequence*.

For each $v \in \mathbf{Z}^k$, define $m(i, v)$ to be the length of the shortest i-bounded, i-covering path in (v, A), if at least one such path exists; if no such path exists, define $m(i, v) = 0$. Now define $f(i) = \max\{m(i, v): v \in \mathbf{Z}^k\}$. (Note that f depends implicitly on the values of A and v_1, which are held fixed.)

For any $v \in \mathbf{Z}^k$, $v_1 \in C(v, A)$ if and only if there is a k-bounded, k-covering path in (v, A). We will obtain an upper bound on $f(i)$ by induction on i; clearly it is only $f(k)$ that we are ultimately interested in.

Remark 3.2. The reader more familiar with Petri nets than vector addition systems will want to think of a vector in $\mathbf{Z}^k$ as a (generalized) marking. A vector being i-bounded or i--r bounded corresponds to a *submarking* being bounded in a particular way.

Lemma 3.3. $f(0) = 1$.

Proof. Trivial. $\square$

Lemma 3.4. $f(i+1) \leq (2^n f(i))^{i+1} + f(i)$ *for* $0 \leq i < k$.

Proof. Let $v \in \mathbf{Z}^k$, let $0 \leq i < k$ be such that there is an $(i+1)$-bounded $(i+1)$-covering path in (v, A).

Case 1: There is an $(i+1)$--$(2^n f(i))$ bounded, $(i+1)$-covering path in (v, A).

Then there must be an $(i+1)$--$(2^n f(i))$ bounded, $(i+1)$-covering path in (v, A) where no two vectors agree on all the first $i+1$ places; the length of such a sequence is $\leq (2^n f(i))^{i+1}$.

Case 2: Otherwise.

Then there is an $(i+1)$-bounded, $(i+1)$-covering path in (v, A) which is *not* $(i+1)$--$(2^n f(i))$ bounded. Then there exist sequences p_1, p_2 such that $p_1 p_2$ is an $(i+1)$-bounded, $(i+1)$-covering path in (v, A), and p_1 is $(i+1)$--$(2^n f(i))$ bounded, and p_2 begins with a vector w which is *not* $(i+1)$--$(2^n f(i))$ bounded; say without loss of generality that $w(i+1) \geq 2^n f(i)$. Clearly, as in case 1, we can choose p_1 to be of length $\leq (2^n f(i))^{i+1}$.

Since p_2 is an i-bounded, i-covering path in (w, A), we know that there exists a path p_2' of length $\leq f(i)$ in (w, A) which is also i-bounded and i-covering. Note now that by definition of the size of a vector, all the places in all the vectors in $A \cup \{v_1\}$ are of absolute value $\leq 2^n$. Since $w(i+1) \geq 2^n f(i)$ and p_2' is of length $\leq f(i)$, we can conclude that p_2' is $(i+1)$-bounded and $(i+1)$-covering. Hence, $p_1 p_2'$ is an $(i+1)$-bounded, $(i+1)$-covering path in (v, A) of length $\leq (2^n f(i))^{i+1} + f(i)$. $\square$

Theorem 3.5. *The covering problem can be decided in space $2^{cn \log n}$ for some constant c.*

Proof. Let (v, A) be a vector addition system of size $\leq n$ and dimension $0<k<n$, and let $v_1 \in \mathbf{N}^k$ be a vector of size $\leq n$. Define $g(0)=2^{3n}$ and $g(i+1)=(g(i))^{3n}$ for $0 \leq i<k<n$; then we see from Lemmas 3.3 and 3.4 that $f(i) \leq g(i)$ for $0 \leq i \leq k$, and hence $f(k) \leq g(k) \leq 2^{(3n)^n} \leq 2^{2^{cn \log n}}$ for some constant c. Hence, we can conclude that $v_1 \in C(v, A)$ if and only if there is some k-bounded, k-covering path in (v, A) of length $\leq 2^{2^{cn \log n}}$ (where c is some constant independent of n). We can compute that the size of any vector on such a path is $\leq 2^{dn \log n}$ for some constant d.

Hence there is a nondeterministic procedure which "guesses" a k-bounded, k-covering path, accepting if and only if one is found, and which operates in space $2^{dn \log n}$. By a well-known theorem of Savitch [8], there is a deterministic algorithm for the covering problem which operates in space $2^{cn \log n}$ for some constant c.

The boundedness problem

Definition 4.1. Let $k \in \mathbf{N}^+$ and let $p = w_1, w_2, \ldots, w_m$ be a sequence of vectors in $\mathbf{Z}^k$, $m>1$. Then p is said to be *self-covering* if $w_j < w_m$ for some j, $1 \leq j \leq m$.

Lemma 4.2 (Karp and Miller [4]). *Let (v, A) be a vector addition system. Then $R(v, A)$ is infinite if and only if there is a k-bounded, self-covering path in (v, A).*

The decision procedure for boundedness suggested by Karp and Miller is as follows: Given (v, A), attempt to enumerate all of $R(v, A)$, and at the same time attempt to find a k-bounded, self-covering path; exactly one of these processes must eventually halt, telling us whether or not $R(v, A)$ is finite. The trouble with this procedure is that if $R(v, A)$ is finite, the size of $R(v, A)$ is not bounded above by a primitive recursive function of the size of (v, A). Therefore their procedure does not operate in primitive recursive time or space. However, we shall exhibit an efficient upper bound on the length of the longest path one must examine in searching for a k-bounded, self-covering path, and in this way obtain an algorithm for boundedness which operates in only exponential space.

It is interesting to note [2, 6] that the problem of determining for two vector addition systems (v, A) and (v', A') of the same dimension, if $R(v, A)$ is finite and equal to $R(v', A')$, can *not* be done in primitive recursive space. Thus, one can decide relatively easily if a reachability set is finite; but to decide if two finite reachability sets are equal, one cannot do much better in the worst case than to enumerate both sets. These results should be compared with the result of Hack [3] that given two systems (v, A) and (v', A') (not necessarily bounded), it is *undecidable* whether or not $R(v, A) = R(v', A')$.

Now let A be a fixed addition set of size $n > 1$ and dimension $k \leq n$.

Definition 4.3. Let $0 \leq i \leq k$. For each $v \in \mathbf{Z}^k$, define $m'(i, v)$ to be the length of the shortest i-bounded, self-covering path in (v, A), if at least one such path exists; if no such path exists, define $m'(i, v) = 0$. Now define $g(i) = \max\{m'(i, v) : v \in \mathbf{Z}^k\}$.

As before, we will obtain an upper bound on $g(i)$ by induction on i, although it is $g(k)$ that we are ultimately interested in. Both the base and induction step are harder than in Section 3, and both will depend on Lemma 4.5. To prove Lemma 4.5, we first need a result of Borosh and Treybis on the size of solutions to linear programming problems.

Lemma 4.4 ([1]). *Let $d_1, d_2 \in \mathbf{N}^+$, let B be a $d_1 \times d_2$ integer matrix and let b be a $d_1 \times 1$ integer matrix. Let $d \geq d_2$ be an upper bound on the absolute values of the integers in B and b. Say that there exists a vector $v \in \mathbf{N}^{d_2}$ which is a solution to the equation set $Bv \geq b$.*

Then for some constant c independent of d, d_1, d_2, there exists a vector $v \in \mathbf{N}^{d_2}$ such that $Bv \geq b$ and such that $v(i) \leq d^{cd_1}$ for all i, $1 \leq i \leq d_2$.

Lemma 4.5. *Let $0 \leq i \leq k$, $v \in \mathbf{Z}^k$, $r > 1$ such that there is an i--r bounded, self-covering path in (v, A). Then there is an i--r bounded, self-covering path in (v, A) of length $\leq r^{n^c}$ for some constant c independent of n, v, r.*

Proof. Let i, v, r be as in the lemma, and consider a minimal length i--r bounded, self-covering path in (v, A); we can write this $v_1, v_2, \ldots, v_{m_0}, w_1, w_2, \ldots, w_{m_1}$ where $w_1 < w_{m_1}$.

For a vector $w \in \mathbf{Z}^k$, define $T(w) \in \mathbf{Z}^i$ by $(T(w))(l) = w(l)$ for $1 \leq l \leq i$. (In case $i = 0$, $T(w)$ is the empty vector, that is, the unique member of $\mathbf{Z}^0$.) Because of our minimality condition, $T(v_1), T(v_2), \ldots, T(v_{m_0})$ must all be distinct. Hence $m_0 \leq r^k$.

The idea of the rest of the proof is as follows: To get a bound on m_1, we will see that the sequence $T(w_1), T(w_2), \ldots, T(w_{m_0})$ can be rearranged to consist, essentially, of a sequence of bounded length, together with (possibly many) simple loops of bounded length hanging off of it; we then use Lemma 4.4 to obtain an upper bound on the number of simple loops necessary to create an i--r bounded, self-covering path.

We now introduce some notation. An *A-sequence* is a finite nonempty sequence of members of A. Let $s = a_1, a_2, \ldots, a_m$ be an A-sequence. By $SUM(s)$ we mean $a_1 + a_2 + \cdots + a_m$. If $w \in \mathbf{Z}^k$, then by $Q(w, s)$ we mean the sequence $w, w + a_1, w + a_1 + a_2, \ldots, w + a_1 + a_2 + \cdots + a_m$; clearly the difference between the first and last vectors of $Q(w, s)$ is Sum(s). If $x \in \mathbf{Z}^i$, then $Q(x, s)$ is the sequence $x, x + T(a_1), x + T(a_1) + T(a_2), \ldots, x + T(a_1) + T(a_2) + \cdots T(a_m)$. If $x \in \mathbf{Z}^i$ and $Q(x, s)$ is i--r bounded, we say that *s is valid for x*.

Say that $x \in \mathbf{Z}^i$, and s is an A-sequence such that $Q(x,s) = x, x_1, \ldots, x_m$ is i--r bounded, $x = x_m$, and $x_{j_1} \neq x_{j_2}$ for any $1 \leq j_1 < j_2 \leq m$; then we call $Q(x, s)$ a *simple x-loop*, or just a *simple loop*, and we call SUM(s) an *x-loop value*, or just a *loop value*.

Note that $T(w_1), T(w_2), \ldots, T(w_{m_1})$ is the projection onto the first i places of an i--r bounded path starting with w_1. If $x, x_1, \ldots, x_m$ is a simple loop occurring as a subsequence of $T(w_1), T(w_2), \ldots, T(w_{m_1})$, then if $x_1, \ldots, x_m$ is removed, the remaining sequence will still be the projection of an i--r bounded path starting with w_1. If this process is repeated carefully, one eventually ends up with a "short" sequence which is the projection of an i--r bounded path starting with w_1, together with a sum $S \in \mathbf{Z}^k$ of loop values for members of sequence; the difference between the first and last members of the i--r bounded path so obtained, plus S, will be equal to $w_{m_1} - w_1 > 0$. Using Lemma 4.4, we obtain a better value for S, and then reverse the above construction by appropriately inserting simple loops in order to obtain the projection of an i--r bounded sequence which starts with w_1, and whose last vector is $> w_1$.

Now define the A-sequence $s_1 = a_1', a_2', \ldots, a_{m_1-1}'$ by $a_j' = w_{j+1} - w_j$ for $1 \leq j < m_1$. Clearly SUM$(s_1) > \bar{0} \in \mathbf{Z}^k$, and s_1 is valid for $T(w_1)$. Let $S_1 = \bar{0} \in \mathbf{Z}^k$. We will define a sequence $s_1, S_1, s_2, S_2, \ldots$ such that for each j:

(1) $Q(T(w_1), s_j)$ and $Q(T(w_1), s_1)$ contain the same set of vectors (with possibly different multiplicities); in particular, s_j is a valid A-sequence for $T(w_1)$.

(2) $S_j \in \mathbf{Z}^k$, and SUM$(s_j) + S_j =$ SUM(s_1).

(3) S_j can be expressed as a nonnegative linear combination of loop values for vectors appearing on $Q(T(w_1), s_j)$.

s_1 and S_1 have already been defined, and satisfy (1), (2), and (3). Assume now that s_j and S_j have been defined, satisfying (1), (2), (3). If the length of s_j is $< (r^k + 1)^2$, then this construction is defined to halt; so assume $s_j = a_1, a_2, \ldots, a_m$ where $m \geq (r^k + 1)^2$, $Q(T(w_1), s_j) = x_1, x_2, \ldots, x_{m+1}$. If we think of the first $(r^k + 1)^2$ members of $x_1, \ldots, x_{m+1}$ being divided up into $(r^k + 1)$ blocks of $r^k + 1$ consecutive vectors, we see that each block contains at least one vector twice, and that in one of the blocks none of the vectors occurs for the first time; this block contains a simple loop, and the removal of this loop from $x_1, \ldots, x_{m+1}$ would not change the total set of vectors appearing. Say that $x_{j_1}, \ldots, x_{j_2}$ is such a simple loop. Define $s_{j+1} = a_1, a_2, \ldots, a_{j_1-1}, a_{j_2}, \ldots, a_m$. Clearly $Q(T(w_1), s_{j+1}) = x_1, x_2, \ldots, x_{j_1-1}, x_{j_2}, \ldots, x_{m+1}$ which has the same set of vectors as those appearing on $Q(T(w_1), s_j)$. Define $S_{j+1} = S_j + a_{j_1} + a_{j_1+1} + \cdots + a_{j_2-1}$, so $S_{j+1} +$ SUM$(s_{j+1}) = S_j +$ SUM(s_j). Since $a_{j_1} + a_{j_1+1} + \cdots + a_{j_2-1}$ is a loop value for x_{j_1} $(= x_{j_2})$, we can conclude that (1), (2), (3) remain true for $j + 1$.

Let us say that this construction ends with S_j, s_j where $s_j = a_1, a_2, \ldots, a_m$, $m < (r^k + 1)^2$. Then $S_j +$ SUM$(s_j) =$ SUM$(s_1) > \bar{0} \in \mathbf{Z}^k$; let $w \in \mathbf{N}^k$ be a vector containing one 1 and the rest 0's such that $S_j +$ SUM$(s_j) \geq w$. Let $L \subseteq \mathbf{Z}^k$ be the set of loop values for vectors occurring on $Q(T(w_1), s_j)$, and let E be the matrix with

k-rows, whose columns are the different members of L. Let $b \in \mathbf{Z}^k$ be the column vector $w - \mathrm{SUM}(s_j)$. Since $S_j \geq w - \mathrm{SUM}(s_j) = b$, and S_j is a nonnegative linear combination of the members of L, the equation system $By_0 \geq b$ has a nonnegative solution, in the integers.

A loop value is just the sum of at most r^k members of A, and so each place of each member of L (and each place of B) is of absolute value $\leq 2^n r^k$. L has therefore at most $(2(2^n r^k)+1)^k$ members, and so B has at most this many columns. Each place of b is of absolute value $\leq 2^n (r^k + 1)^2 + 1$. Letting $d_1 = k$ and $d = r^{3n^2}$, Lemma 4.4 tells us that there is some nonnegative vector y_1 such that $By_1 \geq b$ and such that the sum of the places of y_1 is equal to $l_1 \leq r^{n^c}$ for some constant c.

Now let $s_1' = s_j$ and $S_1' = By_1$. $S_1' \geq w - \mathrm{SUM}(s_1')$, so $S_1' + \mathrm{SUM}(s_1') > \bar{0} \in \mathbf{Z}^k$; in addition S_1' can be expressed as the sum of l_1 loop values of members of $Q(T(w_1), s_1')$. Say that $s_1' = a_1, a_2, \ldots, a_m$ and $Q(T(w_1), s_1') = x_1, x_2, \ldots, x_{m+1}$. Let t be an A-sequence such that $Q(x_{j_1}, t)$ is an x_{j_1} loop for some j_1, $1 \leq j_1 \leq m+1$, and such that $S_1' - \mathrm{SUM}(t)$ can be expressed as a sum of $l_1 - 1$ loop values of members of $Q(T(w_1), s_1')$. Define $S_2' = S_1' - \mathrm{SUM}(t)$ and $s_2' = a_1, a_2, \ldots, a_{j_1-1}, t, a_{j_1}, \ldots, a_m$. Clearly s_2' is valid for $T(w_1)$, and $S_2' + \mathrm{SUM}(s_2') = S_1' + \mathrm{SUM}(s_1') > \bar{0} \in \mathbf{Z}^k$.

Continuing in this way, we construct a sequence $s_1', S_1', s_2', S_2', s_3', S_3', \ldots$, eventually obtaining s', S' such that $S' = \bar{0}$, $\mathrm{SUM}(s') = S_1' + \mathrm{SUM}(s_1') > \bar{0} \in \mathbf{Z}^k$, and s' is valid for $T(w_1)$. The construction takes at most r^{n^c} stages, each stage increasing the length of s_j' by at most r^k.

So $v_1, v_2, \ldots, v_{m_0}, Q(w_1, s')$ is an i--r bounded self-covering path in (v, A), where the length of $Q(w_1, s') \leq r^{n^d}$ for some constant d. So $m_1 \leq r^{n^d}$, $m_0 \leq r^k$, and so $m_0 + m_1 \leq r^{n^c}$ for some constant c. □

Lemma 4.6. *$g(0) \leq 2^{n^c}$ for constant c from Lemma 4.5.*

Proof. Let $v \in \mathbf{Z}^k$ such that there is a self-covering path in (v, A). This is trivially a 0--2 bounded self-covering path, and so Lemma 4.5 tells us there is a 0--2 bounded self-covering path of length $\leq 2^{n^c}$ in (v, A). □

Lemma 4.7. *$g(i+1) \leq (2^n g(i))^{n^c}$ for all i, $1 \leq i < k$, and for constant c from Lemma 4.5.*

Proof. Let $v \in \mathbf{Z}^k$ be such that there is an $(i+1)$-bounded, self-covering path in (v, A). *Case* 1: There is an $(i+1)$--$(2^n g(i))$ bounded, self-covering path in (v, A).

Then by Lemma 4.5, there is an $(i+1)$-bounded self-covering path in (v, A) of length $\leq (2^n g(i))^{n^c}$.

Case 2: Otherwise.

Then there is an $(i+1)$-bounded, self-covering path in (v, A) which is *not* $(i+1)$--$(2^n g(i))$ bounded, call it $w_1, w_2, \ldots, w_m$. Let j, $1 \leq j < m$, be such that $w_j < w_m$. Let l, $1 \leq l < m$, be the smallest number such that w_l is *not* $(i+1)$--$(2^n g(i))$ bounded. Let

$v_1, v_2, \ldots, v_{m_0}$ be a shortest possible path in (v, A) such that v_{m_0} agrees with w_l on the first $i+1$ places, and such that $v_1, v_2, \ldots, v_{m_0-1}$ is an $(i+1)$--$(2^n g(i))$ bounded sequence of vectors. No two of $v_1, v_2, \ldots, v_{m_0-1}$ can agree on the first $i+1$ places, for then the sequence could be made even shorter. Hence, $m_0 - 1 \leq (2^n g(i))^{i+1}$. Without loss of generality, say that $v_{m_0}(i+1) = w_l(i+1) \geq 2^n g(i)$.

For $1 \leq u < m$, define $a_u \in A$ by $a_u = w_{u+1} - w_u$. Using the notation of the proof of Lemma 4.5, let s be the A-sequence $a_l, a_{l+1}, \ldots, a_{m-1}, a_j, a_{j+1}, \ldots, a_{m-1}$. Then $Q(v_{m_0}, s)$ is an $(i+1)$-bounded, and hence i-bounded, self-covering path in (v_{m_0}, A). Let p be an i-bounded, self-covering path in (v_{m_0}, A) of length $\leq g(i)$. Since $v_{m_0}(i+1) \geq 2^n g(i)$ and since each place in each vector in A is at most 2^n in absolute value, p is in fact $(i+1)$-bounded. So $v_1, v_2, \ldots, v_{m_0-1}, p$ is an $(i+1)$-bounded, self-covering path in (v, A) of length $\leq (2^n g(i))^{i+1} + g(i) \leq (2^n g(i))^{n^c}$ (assuming $c > 1$). □

Theorem 4.8. *The boundedness problem can be decided in space $2^{cn \log n}$ for some constant c.*

Proof. Let (v, A) be a vector addition system of size $\leq n$ and dimension k, $0 < k < n$. From Lemma 4.2 $R(v, A)$ is infinite if and only if there is a k-bounded, self-covering path in (v, A); using an analysis like that in the proof of Theorem 3.5, Lemma 4.6 and Lemma 4.7 tell us that such a path exists, if and only if such a path exists of length $\leq 2^{2^{cn \log n}}$ for some constant c. As explained in the proof of Theorem 3.5, there is an algorithm for determining whether such a path exists which operates in space $2^{cn \log n}$ for some constant c. □

5. Conclusion

We have exhibited decision procedures for the covering and boundedness problems for vector addition systems, which operate in exponential space. Recently Sacerdote and Tenney [7] have come up with a decision problem for the *reachability problem* for vector addition systems: given a system (v, A) and a vector $v_0 \in \mathbf{N}^k$, is $v_0 \in R(v, A)$? Their procedure, however, is not primitive recursive since it uses the techniques of [4] used to decide the boundedness problem.

The methods used in Section 4 were sufficient to obtain a primitive recursive upper bound for the boundedness problem; it is an open question, however, if they can be applied with similar effect to the reachability problem.

References

[1] I. Borosh and L. Treybis, Bounds on positive integral solutions of linear diophantine equations, *Proc. AMS* 55 (1976) 199-304.

[2] E. Cardoza, R.J. Lipton and A.R. Meyer, Exponential space complete problems for Petri nets and commutative semigroups, *8th Annual ACM Symp. on Theory of Computing* (1976) 50–54.

[3] M. Hack, The equality problem for vector addition systems is undecidable, *Theor. Comput. Sci.* **2** (1976) 77–96.

[4] R. Karp and R. Miller, Parallel program schemata, *J. Comput. System Sci.* **3** (1969) 147–195.

[5] R. Lipton, The reachability problem requires exponential space, Tech. Report, Dept. of Computer Science, Yale University, New Haven, CN (1975) (to appear in *Theoret. Comput. Sci.*).

[6] E. Mayr, The complexity of the finite containment problem for Petri nets, Tech. Report 181, Laboratory for Computer Science, MIT, Cambridge, MA (1977).

[7] G.S. Sacerdote and R.L. Tenney, The decidability of the reachability problem for vector addition systems, *9th Annual ACM Symp. on Theory of Computing* (1977) 61–76.

[8] W.J. Savitch, Relation between nondeterministic and deterministic tape complexities, *J. Comput System Sci.* **4** (1970) 177–196.