Lecture 2 : Interactive Proofs in EasyCrypt

July 16th, 2013

The Ambient Logic

EasyCrypt ambient logic is a general higher-order logic.

In this talk

- ► How define facts about user defined operators
- ► How to prove them when automatic techniques do not work





Interactive Proofs





EasyCrypt is a typed language:

It comes with a set of core types

unit, bool, int, real, tuple, lists ...

Some of these types are polymorphic (type constructor)

Possibility to create type aliases

type αu $= \alpha * \alpha$ typev= int utypew= int list

Possibility to create abstract types

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Expressions - Functional language

EasyCrypt comes with a functional language:

concrete operators:

op f1 (b : bool) (x y : int) = b ? (x - y) : (x + y). **op** f2 (xs : int list) (x : int) = map (*lambda* (z : int), z + x) xs. **op** f3 (xs : 'a list) = fold (*lambda* v _, v + 1) 0 xs.

► abstract operators:

$$\begin{array}{l} \mathsf{map} : (\alpha \to \beta) \to \alpha \ \textit{list} \to \beta \ \textit{list} \\ \mathsf{fold} \ : (\alpha \to \beta) \to \alpha \ \textit{list} \to \beta \ \textit{list} \end{array}$$

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concrete operators:

abstract operators:

map :
$$(\alpha \rightarrow \beta) \rightarrow \alpha \text{ list } \rightarrow \beta \text{ list}$$

fold : $(\alpha \rightarrow \beta) \rightarrow \alpha \text{ list } \rightarrow \beta \text{ list}$

Predicates / Formulas

Predicates are boolean operators:

op mypred : int \rightarrow int \rightarrow bool.

• These predicates can be defined:

pred mypred (x y : int) = (0 \leq x) \land (0 \leq y) \land (2 * x \leq y)

► Formulas constructors:

$$\begin{array}{ll} \textit{forall} (\mathsf{x}:\mathsf{t}), \phi & (\forall (\mathsf{x}:\mathsf{t}), \phi) \\ \phi_1 / \langle \phi_2 & (\phi_1 \land \phi_2) \\ \phi_1 = > \phi_2 & (\phi_1 \Rightarrow \phi_2) \\ ! \phi & (\neg \phi) \end{array}$$

exists (x : t),
$$\phi$$
(\exists (x : t), ϕ) $\phi_1 \setminus / \phi_2$ ($\phi_1 \lor \phi_2$) $\phi_1 <=> \phi_2$ ($\phi_1 \Leftrightarrow \phi_2$)+ dedicated formulas for p(R)HL

Axioms / Lemmas

Formulas for operators axiomatization:

op count : 'a list -> int.

• Formulas for stating facts:

lemma fact (x y : int): $x \le 0 \rightarrow y \le 0 \rightarrow 0 \le x * y$.

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Stating a theorem





Progress is done via tactics that allows the *simplification*, *decomposition* into *subgoals*, or the *resolution* of the goal.

Stating a theorem



$$\underbrace{(b1 \Rightarrow b2) \Rightarrow (b2 \Rightarrow b3) \Rightarrow b1}_{\text{assumptions}} \Rightarrow \underbrace{b3}_{\text{conclusion}} \ \} \text{ goal}$$

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lemma mylemma b1 b2 b3 : ... proof. intros⇒ hb12.

> b1 : bool b2 : bool b3 : bool hb12 : b1 \Rightarrow b2

$$(b2 \Rightarrow b3) \Rightarrow b1 \Rightarrow b3$$

```
lemma mylemma b1 b2 b3 : ...
proof.
intros⇒ hb12 bh23 hb1.
```

b1 : bool b2 : bool b3 : bool hb12 : b1 \Rightarrow b2 hb23 : b2 \Rightarrow b3 hb1 : b1

b3

```
lemma mylemma b1 b2 b3 : ...
proof.
intros⇒ hb12 bh23 hb1.
apply hb23.
```

 $\begin{array}{l} \texttt{b1:bool}\\ \texttt{b2:bool}\\ \texttt{b3:bool}\\ \texttt{hb12:b1} \Rightarrow \texttt{b2}\\ \texttt{hb23:b2} \Rightarrow \texttt{b3}\\ \texttt{hb1:b1} \end{array}$

```
lemma mylemma b1 b2 b3 : ...
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intros⇒ hb12 bh23 hb1.
apply hb23.
apply hb12.
```

```
\begin{array}{l} \texttt{b1:bool}\\ \texttt{b2:bool}\\ \texttt{b3:bool}\\ \texttt{hb12:b1} \Rightarrow \texttt{b2}\\ \texttt{hb23:b2} \Rightarrow \texttt{b3}\\ \texttt{hb1:b1} \end{array}
```

```
lemma mylemma b1 b2 b3 : ...
proof.
intros⇒ hb12 bh23 hb1.
apply hb23.
apply hb12.
assumption.
```

Proof completed

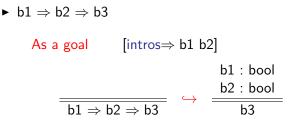
```
lemma mylemma b1 b2 b3 : ...
proof.
intros⇒ hb12 bh23 hb1.
apply hb23.
apply hb12.
assumption.
qed.
```

Propositional logic

 $\blacktriangleright \ b1 \Rightarrow b2 \Rightarrow b3$

 $\begin{array}{ll} \mbox{As a goal} & [\mbox{intros} \Rightarrow b1 \ b2] \\ \mbox{As an hypothesis} & [\mbox{apply}] \end{array}$

Propositional logic



As an hypothesis [apply]

Propositional logic

► $b1 \Rightarrow b2 \Rightarrow b3$ As a goal [intros \Rightarrow b1 b2] As an hypothesis [apply] $\mathsf{h}:\mathsf{b1} \xrightarrow{\Rightarrow} \mathsf{b2} \xrightarrow{\Rightarrow} \mathsf{b3} \quad \hookrightarrow$ b3 1. b1 b2

• Conjunction: $a \land b$

As a goal [split] (prove $a \land b$)

As an hypothesis [elim ab] (destruct $a \land b$ in a and b)

► Conjunction: a ∧ b

As a goal [split] (prove $a \land b$) $a \land b \land b \land b$ $(a \land b) \land b \land b$

As an hypothesis [elim ab] (destruct $a \land b$ in a and b)

As a goal [split] (prove $a \land b$) = $a \land b$ \hookrightarrow 1. = a 2. = b

As an hypothesis [elim ab] (destruct a \land b in a and b)

 \blacktriangleright Disjunction: a \lor b

As a goal $\label{eq:asymptotic} \mbox{As an hypothesis} \qquad \mbox{[elim ab]} \qquad \mbox{(case analysis on a \lor b)}$

• Disjunction: a \lor b

As a goal

• [left] (prove a \lor b by proving a)

$$a \lor b$$
 a

• [right] (prove a \lor b by proving b)

$$a \lor b \qquad b$$

As an hypothesis [elim ab] (case analysis on a \lor b)

• Disjunction: $a \lor b$

As a goal ● [left] (prove a ∨ b by proving a)

$$a \lor b$$
 a

• [right] (prove $a \lor b$ by proving b) $a \lor b \hookrightarrow b$ As an hypothesis [elim ab] (case analysis on $a \lor b$) $ab : a \lor b \\ \phi \to 1$. $a \Rightarrow \phi$ 2. $b \Rightarrow \phi$

Propositional logic - existential

• Existential: *exists* $x : t, \phi(x)$

As a goal [*exists* v] (prove goal by giving a witness)

As an hypothesis [elim h] (extract a witness)

Propositional logic - existential

• Existential: *exists* $x : t, \phi(x)$

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$$\underbrace{=} exists x : t, \phi(x) \qquad \longrightarrow \qquad \underbrace{=} \phi(v)$$

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Propositional logic - existential

• Existential: *exists*
$$x : t, \phi(x)$$

As a goal [exists v] (prove goal by giving a witness)

$$\underbrace{exists \, \mathsf{x} : \mathsf{t}, \phi(\mathsf{x})}_{exists \, \mathsf{x} : \mathsf{t}, \phi(\mathsf{x})} \quad \hookrightarrow \quad \underbrace{\phi(\mathsf{v})}_{\phi(\mathsf{v})}$$

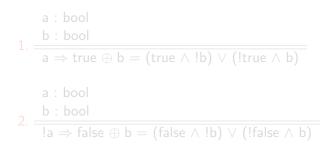
As an hypothesis[elim h](extract a witness) $h : exists \times : t, \phi(x)$ \hookrightarrow $\overleftarrow{forall (v : t), \phi(v) \Rightarrow \phi'}$

Boolean case analysis

The tactic case allows to do a case analysis on any formula.

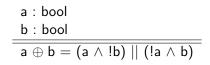


(case a) leads to



Boolean case analysis

The tactic case allows to do a case analysis on any formula.



(case a) leads to a : bool 1. $\frac{b : bool}{a \Rightarrow true \oplus b = (true \land !b) \lor (!true \land b)}$ a : bool 2. $\frac{b : bool}{!a \Rightarrow false \oplus b = (false \land !b) \lor (!false \land b)}$

Boolean case analysis

The tactic case allows to do a case analysis on any formula.

$$\begin{array}{c} \mathsf{a} : \mathsf{bool} \\ \mathsf{b} : \mathsf{bool} \\ \end{array} \\ \hline \mathsf{a} \oplus \mathsf{b} = (\mathsf{a} \land !\mathsf{b}) \mid\mid (!\mathsf{a} \land \mathsf{b}) \\ \end{array}$$

(case a) leads to

a : bool 1. b : boola \Rightarrow true \oplus b = (true \land !b) \lor (!true \land b) a : bool 2. b : bool!a \Rightarrow false \oplus b = (false \land !b) \lor (!false \land b)

Identification up to computations

EasyCrypt comes with a set of simplification rules.

a : bool b : bool false \oplus b = (false \land !b) \lor (!false \land b)

simplify leads to

a : bool b : bool b = b

that can be easily solved by reflexivity.

Identification up to computations

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simplify leads to

$$a : bool$$

b : bool
b = b

that can be easily solved by reflexivity.

Identification up to computations

Computations include

- functions applications reduction
- operators body inlining
- ► logical operators tautology (a \land false \rightarrow false)

Terms that are equal up to computations are considered as identical

- a : bool
- b : bool

 $!a \Rightarrow \mathsf{false} \oplus \mathsf{b} = (\mathsf{false} \land !\mathsf{b}) \lor (!\mathsf{false} \land \mathsf{b})$

can be directly solved by reflexivity.

The tactic rewrite replaces a subterm a of the goal by an equal one b. It takes a proof of a = b or $a \Leftrightarrow b$.

 $\begin{array}{c} \text{rewrite h} \\ \hline h:a=b \\ \hline Pa \end{array} \xrightarrow{} \begin{array}{c} h:a=b \\ \hline Pb \end{array}$

The tactic rewrite replaces a subterm a of the goal by an equal one b. It takes a proof of a = b or $a \Leftrightarrow b$.

rewrite h $\frac{h:a=b}{Pa} \leftrightarrow \frac{h:a=b}{Pb}$

- ▶ rewrite -h : from right to left
- rewrite mh where m is a multiplier
 - ? as many times as possible
 - ! as many times as possible, at least one
 - n? at most n times
 - n! exactly n times
- rewrite {o}h where o is a sequence of positive integers
 Rewrites the oth occurrences only.

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$$2 * (a + b) = (b + a) + (a + b)$$

► rewrite {2}addnC

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rewrite (addnC b a)
 2 * (a + b) = (a + b) + (a + b)

▶ rewrite -!addnA

$$2 * (a + b) = b + (a + (a + b))$$

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Logical cut

The tactic $\operatorname{cut:} \phi$ allows to do a forward chaining

$$\frac{\mathbf{h}: \dots}{\phi'} \quad \hookrightarrow \quad \mathbf{1}. \quad \frac{\mathbf{h}: \dots}{\phi} \quad \mathbf{2}. \quad \frac{\mathbf{h}: \dots}{\phi \Rightarrow \phi'}$$

It is possible to give a name to the new goal (cut my: ϕ)

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It is possible to give a name to the new goal (cut my: ϕ)

An induction principle for a type t is any formula of the form:

$$\forall (p: t \rightarrow bool), \phi_1 \rightarrow ... \rightarrow \phi_n, \forall (x: t), psi1(x) \rightarrow ... \rightarrow psin(x) \rightarrow p x$$

For example, for natural numbers:

$$\begin{array}{l} \textit{forall (p:int \rightarrow t), p 0 \Rightarrow} \\ (\textit{forall (x:int), 0 \leq x \Rightarrow p x \Rightarrow p (x + 1)) \Rightarrow} \\ \textit{forall (x:int), 0 \leq x \Rightarrow p x} \end{array}$$

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For example, for natural numbers:

Applying the induction principle via apply can be cumbersome.

The tactic elimT eases the applications of such principles.

$$\begin{array}{ccc} P: \text{ int } \to \text{ bool} \\ \hline 0 \leq x \Rightarrow P x \end{array} \quad \hookrightarrow \quad \text{elimT ind } x \end{array}$$

1.
$$\frac{\mathsf{P}:\mathsf{int}\to\mathsf{bool}}{\mathsf{P}\;\mathsf{0}}$$
 2.
$$\frac{\mathsf{P}:\mathsf{int}\to\mathsf{bool}}{\forall(\mathsf{x}:\mathsf{int}),\,\mathsf{0}\leq\mathsf{x}\to\mathsf{P}\;\mathsf{x}\to\mathsf{P}\;(\mathsf{x+1})}$$

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Automation

EasyCrypt comes with some automation tactics:

- progress break the goal by repetead applications of the introduction based tactics (split, intros, ...)
- ► trivial: same as progress, but try to close subgoals.
- smt: try to solve the goal calling external SMT solvers.

Plan

The EasyCrypt Core Language

Interactive Proofs





Tacticals are operators on tactics.

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- t1; t2 apply t1 and then t2 on all generated subgoals
- ▶ t; [t1|...|tn]

apply t and then each of the t_i to the i^{th} subgoal

► do t

repeat t as much as possible, at least one time this tactic takes the same multiplier of rewrite do! t, do? t, do n! e, do n? t

► try t

try to apply t, or nothing if t cannot by applied

▶ by t1; ...; tn

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Tacticals are operators on tactics.

► t1; first t2

apply t1 and then t2 on the first subgoal

► t1; last t2

apply t1 and then t2 on the last subgoal

- variants: t1; first n t2, t1; last n t2
- ▶ t; first n last

apply t and then shift the n first goals to the end

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Tacticals - Intros

Tacticals are operators on tactics.

 $\blacktriangleright \ t \Rightarrow ip1 \ ... \ ipn$

apply t and then execute the introduction of $\mathsf{ip1}\ \ldots\ \mathsf{ipn}$

• t \Rightarrow x introduce a name / an hypothesis

• $t \Rightarrow [ip1|...ipn]$

execute ip; on the *i*th subgoal

+ do a case analysis if not done by t

• t \Rightarrow \rightarrow

introduce an equational hypothesis and rewrite it

• $t \Rightarrow \{h\}$

clear the hypothesis h

• t \Rightarrow //

execute trivial

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$$t \Rightarrow [ip1|...ipn]$$

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$$t \Rightarrow \{h\}$$

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execute trivial

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The EasyCrypt Core Language

Interactive Proofs





trade-off between interactive / automatic proof

EasyCrypt has now two kinds of tactics

- low-level, interactive ones
- the SMT hammer

The difficulty is to find the right trade-off between the two.

- SMT goal resolution success can be very unstable
- SMT can be very good in solving large or numerous problems generated by p(R)HL judgments
- qed does not mark the end of the proof.