A Logic for Information Flow in Object-Oriented Programs *

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Abstract

This paper specifies, via a Hoare-like logic, an interprocedural and flow sensitive (but termination insensitive) information flow analysis for object-oriented programs.

Pointer aliasing is ubiquitous in such programs, and can potentially leak confidential information. Thus the logic employs *agreement assertions* to describe the noninterference property that formalizes confidentiality, and employs *points-to assertions* to describe possible aliasing. Programmer assertions, in the style of JML, are allowed, thereby permitting a more fine-grained specification of information flow analysis.

The logic supports local reasoning about state in the style of Separation Logic. Small specifications are used; they mention only the variables and addresses relevant to a command. Specifications are combined using a frame rule. An algorithm for the computation of postconditions is described: under certain assumptions, there exists a *strongest* postcondition which the algorithm computes.

1 Introduction

An information flow policy, concerned with protecting confidentiality of data, must ensure that during program execution, data does not flow to a channel unauthorized to receive the data [9]. The typical setting for checking confidentiality of data involves channels with different clearance levels[1], e.g., High for sensitive/private channels and Low for public channels, and a program that manipulates data arriving at input channels (with different clearance levels) and produces results that may flow into output channels (with different clearance levels). In this setting, confidentiality of data can be assured provided that, during program execution, data meant for High output channels do not flow into Low output channels. Cohen [13] advanced an equivalent, deductive formulation for assuring confidentiality: from the text of the program, and by observing only the data in Low output channels (hereafter called Low outputs) an attacker cannot deduce any information about the data in High input channels (hereafter called High inputs). In other words, for confidentiality to hold, Low outputs must not depend on High inputs in any way. It is this notion of *independence* that is explored in this paper in the context of object-oriented programs.

*This paper is technical report KSU CIS-TR-2005-1. A few modifications, like replacing the terms "independence assertion" and "region assertion" by "agreement assertion" and "points-to assertion", were made in August 2006. Section 7, on implementing the logic, is in a quite rough state, and will soon be superseded by a paper devoted to the automatic inference of assertions.

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[1] In general, these levels form a security lattice, with Low $\leq$ High.

Here are some simple examples that illustrate whether or not a program satisfies confidentiality. In each example, the variable l is a Low output and the variable h is a High input. First, the assignment $l := h$ violates confidentiality directly due to the data flow from h to l. Second, the conditional **if** $h > 0$ **then** $l := 1$ **else** $l := 0$ violates confidentiality indirectly due to control flow: while neither assignment by itself violates confidentiality, information as to whether or not $h > 0$ is revealed by whether or not l is 1 after the execution. In contrast, the command $l := h\,; l := 0$ satisfies confidentiality although it has a subpart that does not: no deductions can be made about the input value of h from the output value of l, since the latter is always 0.

Information flow analysis has been used to statically certify[14] that confidentiality holds in all possible execution paths of a program. Typical information flow analyses, surveyed by Sabelfeld and Myers [23], are often specified using security type systems [25, 18, 21, 5, 17]. The security guarantee provided by a well-typed program is this: no High inputs will flow to Low outputs either directly, via data flow, or indirectly, via control flow, during program execution. The type systems mentioned above, except for the recent [17], are flow insensitive, and this is a source of imprecision. Indeed, such type systems reject all the example programs above, including the benign one, for they require every subprogram be well-typed whether or not it contributes to the final answer. The subprogram, $l := h$, in the benign example, fails to type.

Extant security type systems for object-oriented programs [5, 18] have yet another source of imprecision that arises due to the way aliasing is handled. In object-oriented programs, fields of a class – in addition to program variables – are annotated with security levels. However, if an object is assigned to a High variable, then the Low fields of the object cannot be updated [5, 18]. Thus the field update, $z.\mathit{info} := 42$, is rejected by the security type system in case info has level Low and z has level High. The reasoning is as follows: consider two Low variables, p and q, which are assigned objects o_1 and o_2 respectively. Now consider the command **if** $h > 7$ **then** $z := p$ **else** $z := q$ which appears secure since a High variable is updated under a High guard. However, depending on h, either z and p are aliases of o_1, or z and q are aliases of o_2. A subsequent update of z's info field will reveal information about h: if $q.\mathit{info}$ is not 42 after the field update, we know that $h > 7$ holds. A similar reasoning requires a method call like $x.m(y)$ to update only High fields in the body of method m, in case the receiver x is High. Such reasoning, while sound, is imprecise: aliasing may not be present at all, in which case, both the field update and the method call is benign.

Our challenges are twofold. First, we prefer a flow sensitive specification of information flow analysis. We also want to handle pointer aliasing in a manner that is more precise than extant approaches which do not perform any alias analysis.

The second challenge is to obtain a *modular* specification for an interprocedural information flow analysis. (Ideally, this would allow us to obtain a static checker for information flow). To be specific, we want our analysis to be compositional in the *state*.[2] We want local reasoning about the *heap* where aliasing happens; this means that when we analyze a command, we are only allowed to consider the *footprint* of the command on the state, i.e., we can only consider the variables and parts of the heap that are used by the command [20, 22] – nothing else.

Contributions. The primary contribution of this paper is to meet both of the above challenges by specifying an interprocedural information flow analysis using a Hoare-like logic. Assertions in the logic are stateful and describe aliasing properties – *points-to assertions* – as well as information flow properties – *agreement assertions*. To reason about outgoing method calls in method bodies, we require method summaries to provide a contract about assertions that must be met before a call and assertions that must hold after a call.

Importantly, the logic uses fundamental ideas from separation logic [20, 22] to provide local reasoning about state. As we clarify in the sequel, specifications in the logic are *small* or *local*: the intuition is that these specifications convey the "bare essence" of reasoning about a command. The reasoning can be elaborated in different contexts, and larger specifications may be obtained by way of a *frame rule*. Indeed, with points-to and agreement assertions, our specification yields an interprocedural static checker for information flow.

[2] It is not compositional reasoning per se we are interested in, since it is "perfectly possible to be compositional and global (in the state) at the same time, as was the case in early denotational models of imperative languages" [20].

Our second contribution is to extend the logic with programmer assertions so that a more fine-grained specification of information flow policy can be obtained. Programmer assertions can take the form "x is a constant", or "variables x and y are equal", or "$x = k(y)$", where k is a mathematical function: such assertions are also allowed, e.g., in JML [11]. In contrast to points-to and agreement assertions, however, programmer assertions may require runtime checking or verification by a theorem prover. We show examples of the use of programmer assertions in concert with points-to and agreement assertions for verifying observational purity [6] and for demonstrating selective dependency [13]. Nevertheless, we do not have an automatic checker in the presence of programmer assertions. At some points in the checking process, "logical implications" need to be decided. We do not know whether there exists a useful proof system to decide the logical implications. But we provide a few simple heuristics to ease the burden of checking.

A minor contribution of the paper is concerned with completeness issues for the logic with assertions restricted to points-to and agreement assertions only. For this sub-logic, we give an algorithm that computes postconditions from preconditions and show that, under certain extra assumptions, the sub-logic is complete: there exists a *strongest* postcondition that the algorithm computes[3]. Alas, the algorithm is non-modular. The main difficulty lies with interprocedural analysis, for which the procedure summaries must be discovered and updated on the fly. We leave this issue for a future paper.

2 Examples

Local Reasoning about Aliasing. Recall that local reasoning about a command entails reasoning only about the *footprint* of the command. In the command, $z.\mathit{info} := 42$, for example, reasoning is permitted only with variable z, the location in the heap that z denotes, and the contents of the *info* field – nothing else. Since we are interested in static checking, we need to abstract the concrete heap location denoted by z.

Abstract locations (as in, e.g., [19]) are used to abstract sets of concrete heap locations. A points-to assertion $x \rightsquigarrow L$, read "x at L", asserts that L abstracts the concrete location denoted by x.

Suppose two abstract locations L_1 and L_2 are disjoint, i.e., they abstract two disjoint sets of concrete locations. Then, if $x \rightsquigarrow L_1$ and $y \rightsquigarrow L_2$ hold, we infer that x, y *must not* alias a concrete location. (In contrast, if L_1, L_2 are not disjoint, then x, y *may* alias).

Points-to assertions may also take the form $L_1.f \rightsquigarrow L_2$, so as to deal with aliasing caused by heap-allocated values, e.g, $x.f$. The intuition is that for any concrete location ℓ_1 that is abstracted by L_1, if field f of ℓ_1 contains concrete location ℓ_2, then ℓ_2 is abstracted by L_2.

We now show two examples in which points-to assertions are used to reason locally about aliasing. Consider a method *getNode* which, given the head of a linked list and an integer i, returns the node at position i in the list. Each node has two fields: *data* denoting the value in the node, and *next* denoting the next node in the list. We consider two implementations of *getNode*: in the first, a pointer to the ith node is returned, creating an alias; in the second, a copy of the ith node is returned – this does not create an alias. The bodies of *getNode* for the two implementations are shown below; the distinguished variable, *result*, holds the return result of a method.

$$
\begin{array}{l}
n := \mathit{head};\ j := 0;\\
\mathbf{while}\ (n \neq \mathbf{null})\ \&\&\ (j < i)\ \mathbf{do}\\
\quad \{n := n.\mathit{next};\ j := j + 1;\}\\
\mathit{result} := n
\end{array}
$$

[3] By "strongest" postcondition we mean the strongest among the assertions accepted by our logic, rather than the strongest among the assertions which are "semantically correct" (a larger set).

Example 1: Node i is aliased

$$
\begin{array}{l}
n := \mathit{head};\ j := 0; \\
\textbf{while}\ (n \neq \textbf{null})\ \&\&\ (j < i)\ \textbf{do} \\
\quad \{n := n.\mathit{next};\ j := j + 1;\} \\
\textbf{if}\ n \neq \textbf{null then} \\
\quad \{\mathit{newNode} := \textbf{new}\ \mathit{Node}; \\
\quad \mathit{newNode.data} := n.\mathit{data};\ \mathit{newNode.next} := \textbf{null}; \\
\quad \mathit{result} := \mathit{newNode};\} \\
\textbf{else}\ \{\mathit{result} := \textbf{null}\}
\end{array}
$$

Example 2: Node i is not aliased

Consider the first two commands of Example 1, where we assume that L is the abstract location in which the list is allocated. Because *head* points to the first node in the list, $\mathit{head} \rightsquigarrow L$ is part of the precondition of the program, which also contains the assertion $L.\mathit{next} \rightsquigarrow L$. For the command $n := \mathit{head}$, we get the small specification:

$$\{\mathit{head} \rightsquigarrow L\}\ n := \mathit{head}\ \{n \rightsquigarrow L\}$$

The specification says that from precondition $\mathit{head} \rightsquigarrow L$, the postcondition $n \rightsquigarrow L$ can be asserted. Note how the points-to assertions in the specification mention facts about *head* and n, nothing else. Next, for the command $j := 0$, we get the small specification[4] $\{\mathit{true}\}\ j := 0\ \{j \rightsquigarrow \textbf{int}\}$. To combine the specifications for the two commands above, we use, in a manner similar to separation logic, a frame rule (also see [10]): because n is not modified by $j := 0$, the frame rule allows us to add $n \rightsquigarrow L$ as conjunct to both its pre- and postconditions. To wit:

$$\{n \rightsquigarrow L\}\ j := 0\ \{j \rightsquigarrow \textbf{int}, n \rightsquigarrow L\}$$

Now the two specifications can be combined to obtain the following specification for the sequential composition, $n := \mathit{head};\ j := 0$.

$$\{\mathit{head} \rightsquigarrow L\}\ n := \mathit{head};\ j := 0\ \{j \rightsquigarrow \textbf{int},\ n \rightsquigarrow L\}$$

The invariant for the while loop is $\{n \rightsquigarrow L, L.\mathit{next} \rightsquigarrow L\}$, which we may write in abbreviated form as $\{(n, L.\mathit{next}) \rightsquigarrow L\}$. To show that the preamble establishes this invariant from the program's precondition, we may apply the frame rule once more on the above specification, adding $L.\mathit{next} \rightsquigarrow L$ to both pre- and postcondition; this is valid since no *next* field is modified by the preamble. Thus:

$$\{(\mathit{head}, L.\mathit{next}) \rightsquigarrow L\}\ n := \mathit{head};\ j := 0\ \{(n, L.\mathit{next}) \rightsquigarrow L\}$$

To show that the invariant is maintained by the while loop, we show the stronger property that each assignment in the loop body maintains the invariant. For $n := n.\mathit{next}$ the small specification is $\{(n, L.\mathit{next}) \rightsquigarrow L\}\ n := n.\mathit{next}\ \{n \rightsquigarrow L\}$. Now the frame rule (applicable since no *next* field is modified) gives us

$$\{(n, L.\mathit{next}) \rightsquigarrow L\}\ n := n.\mathit{next}\ \{(n, L.\mathit{next}) \rightsquigarrow L\}$$

In a similar (but simpler) way, we can show $\{(n, L.\mathit{next}) \rightsquigarrow L\}\ j := j + 1\ \{(n, L.\mathit{next}) \rightsquigarrow L\}$. Finally, for $\mathit{result} := n$, the small specification is

$$\{n \rightsquigarrow L\}\ \mathit{result} := n\ \{\mathit{result} \rightsquigarrow L\}$$

By a few more applications of the frame rule, we obtain the following specification for the body, B_1, of *getNode*.

$$\{(\mathit{head}, L.\mathit{next}) \rightsquigarrow L\}\ B_1\ \{(n, L.\mathit{next}, \mathit{result}) \rightsquigarrow L\}$$

[4] The assertion $j \rightsquigarrow \textbf{int}$, expressing that j has an integer value, is strictly speaking redundant, since we shall assume that we are dealing with "well-typed" programs where a variable/field may contain an integer iff it has been assigned the type **int**. Therefore such assertions may be omitted.

As expected, n and *result may alias* the same location in the heap.

In Example 2, the precondition for the entire method body, B_2, of *getNode* is the same as that of B_1, namely, $(head, L.next) \rightsquigarrow L$. The crucial difference is the occurrence of the command $newNode := \textbf{new}\ Node$ where we may choose an arbitrary abstract location to abstract the concrete location being created. Choosing L_1, we get the small specification

$$\{true\}\ newNode := \textbf{new}\ Node\ \{newNode \rightsquigarrow L_1\}\ .$$

Applying the frame rule repeatedly, we can derive postcondition[5]

$$\{(n, L.next) \rightsquigarrow L, (result, newNode) \rightsquigarrow L_1, L_1.next \rightsquigarrow \bot\}$$

for B_2. The key observation is that provided L and L_1 are disjoint, n and *result must not alias* the same location in the heap.

Information Flow Analysis and Independences. A baseline correctness property for information flow analysis is noninterference [16] (the negation of Cohen's notion of dependency [13]) which is formalized via an "indistinguishability" relation on states. Two states are indistinguishable if they agree on values of their Low variables (but may differ on values of High variables). Noninterference holds if any two runs of a program starting in two initially indistinguishable states, yield two final states that are also indistinguishable. In other words, a program is noninterfering, if for any pair of runs, changes to its High input variables are unobservable via its Low output variables; hence, reverting to a point made in the introduction, Low outputs are *independent* of High inputs.

The small specifications of our analysis are designed to answer the following question, encompassing noninterference as a special case[6]: *given two runs which initially agree on variables $x_1 \dots x_n$, will they at the end agree on variables $y_1 \dots y_m$?* Accordingly, we introduce *agreement assertions* of the form $x\ltimes$, such that a positive answer to the above question amounts to the specification $\{x_1\ltimes, \dots, x_n\ltimes\}\ \_\ \{y_1\ltimes, \dots, y_m\ltimes\}$. In general, we shall consider assertions of the form $a\ltimes$, where a is an *abstract address*: either a variable, or a field access of the form $L.f$.

Leveraging the above reading of noninterference, Amtoft and Banerjee specified, as a Hoare-like logic, a termination insensitive information flow analysis for simple imperative programs [1] (later extended to a termination *sensitive* analysis [2]). This paper extends that logic to handle programs written in a core, Java-like, object-oriented language. Also, unlike [1, 2], this paper employs a standard style semantics.

Aliasing, Independences and Local Reasoning. We consider the following example adapted from Askarov's master's thesis [3].

$$\begin{array}{l}\textbf{class}\ X\{\\ \quad \textbf{int}\ q;\\ \quad \textbf{int}\ getQ()\{result := self.q\};\\ \quad \textbf{unit}\ setQ(\textbf{int}\ n)\{self.q := n\}\}\end{array}$$

What can we say about the body of *getQ*? First, we consider points-to assertions. Suppose assertions $self \rightsquigarrow \rho_1$ and $\rho_1.q \rightsquigarrow \textbf{int}$ hold for the precondition of *getQ*. Then we can assert that $result \rightsquigarrow \textbf{int}$ holds in the postcondition of *getQ*. Think about ρ_1 as a metavariable which will be instantiated by abstract locations at the point of call. For instance, if the receiver in the call to *getQ* is at abstract location L, then ρ_1 will be substituted by L.

Next, we consider agreement assertions. Given that $self \rightsquigarrow \rho_1$ holds for the precondition of *getQ*, we want to check whether the postcondition contains $result\ltimes$. That is, under which conditions will two runs agree on the final value of *result*? For that to be the case, the runs must agree on the initial value of $self.q$,

[5] The abstract location $\bot$ abstracts null pointers only.

[6] As can be seen by letting $x_1 \dots x_n$, and $y_1 \dots y_m$, be the Low variables.

a sufficient condition for which is that $\rho_1.q\ltimes$ holds in the precondition; also (since *self*$.q$ depends on *self*), the runs must agree on *self*. A convenient *method summary* for *getQ* is thus the following

$$\{self \rightsquigarrow \rho_1, self\ltimes, \rho_1.q\ltimes\}\ getQ\ \{result\ltimes\}\ .$$

On the other hand, if the agreement assertions in the precondition do not hold at the point of call, we are unable to conclude *result*$\ltimes$ in the postcondition.

In a similar manner, we can compute the following method summary for *setQ*:

$$\{self \rightsquigarrow \rho_1, self\ltimes, n\ltimes, \rho_1.q\ltimes\}\ setQ\ \{\rho_1.q\ltimes\}$$

This says that in order for two runs to agree on the final value of the q fields of "corresponding" (as formalized in Sec. 4) objects abstracted by ρ_1, they must agree on the initial value of n, and on the initial value of *self* (as otherwise, the two runs would update non-corresponding objects). Also, because there may be other objects abstracted by ρ_1 than the one which *self* points to (and these objects did not have their q field updated), the runs must agree on the initial value of all q fields; this requirement can be omitted in the case where ρ_1 abstracts one concrete location only, i.e., in the case of "strong update".

Now consider the program

$$\begin{array}{l} X\ x_1;\ \ X\ x_2 := \textbf{new}\ X; \\ x_1 := x_2; \text{//alias created} \\ x_1.setQ(secret); \\ z := x_2.getQ() \end{array}$$

where, because x_1 and x_2 are aliases, the value of *secret* is leaked to z. Let us see how checking independences might help detect the leak. We recall what noninterference means: two runs that initially agree on all variables except for *secret*, must agree on the final value of z. A proof of noninterference, in our framework, would thus amount to establishing a specification where $z\ltimes$ is in the postcondition, *without* having to assume that *secret*$\ltimes$ is in the precondition. Below, we argue that this is impossible.

First assume that the location allocated by **new** is abstracted by L_2; then we have $x_2 \rightsquigarrow L_2$ and $x_1 \rightsquigarrow L_2$. With the aim of proving that $z\ltimes$ holds after the call to *getQ*, we consult the method summary for *getQ* where we substitute *self* by x_2, and *result* by z, and ρ_1 by L_2. Looking at the resulting precondition, we see that we must show that $x_2\ltimes$ and $L_2.q\ltimes$ holds before the call to *getQ*, that is, after the call to *setQ*. We therefore consult the summary for *setQ* where we substitute *self* by x_1, n by *secret*, and ρ_1 by L_2. Looking at the resulting precondition, we see that we must at least show that *secret*$\ltimes$ holds. But this yields the desired contradiction.

Suppose the aliasing were removed in a slight modification of the above program, where z is once again the output variable:

$$\begin{array}{l} X\ x_1 := \textbf{new}\ X;\ \ X\ x_2 := \textbf{new}\ X; \text{//no alias} \\ x_1.setQ(secret);\ \ z := x_2.getQ() \end{array}$$

Now x_1 and x_2 do not alias the same heap location. The postcondition for the first assignment asserts $\{x_1 \rightsquigarrow L_1, x_1\ltimes\}$, and that for the second asserts $\{x_2 \rightsquigarrow L_2, x_2\ltimes\}$, where L_1 and L_2 are assumed disjoint to reflect the absence of aliasing. As before, to establish that $z\ltimes$ holds after the call to *getQ*, we must show that $x_2\ltimes$ and $L_2.q\ltimes$ holds after the call to *setQ*. But since locations abstracted by L_2 are not modified by the call to *setQ*, this follows from the frame rule (since we may assume that $L_2.q\ltimes$ holds before the call). In summary, because of the absence of aliasing, the assertion $z\ltimes$ does hold finally, even if *secret*$\ltimes$ does not hold initially. This is in contrast to the previous example.

It is instructive to see how an existing type-based information flow analysis system, like Jif [18], handles the above programs. Assume that the variables *secret* and x_1 are typed High, and x_2 and field q are typed Low. Since q is Low, the method *setQ* has a *begin label* of Low, which says that the method can only be called if the program counter of the caller is no more restrictive than Low. But the level of the receiver (x_1) is High. This is one reason why Jif rejects this program. In general, the above check ensures that if there are any low aliases of x_1 in the future – e.g., x_2 in the first program – they should not be able to read the value of q assigned by *setQ*. In the second example there is no aliasing. Yet, Jif rejects this example also, because the call to *setQ* is untypable.

Programmer assertions. As noted earlier, apart from points-to assertions and agreement assertions, we also allow programmer assertions in code. For the trivial program **if** $x > 0$ **then** $w := 7$ **else** $w := 7$, e.g., clearly $w\ltimes$ holds (two runs will *always* agree on the final value of w), although a naïve analysis cannot prove the assertion. However, armed with the programmer assertion that w is a specific constant after the conditional, the following reasoning is sound in our framework: w being constant "logically implies" (defined in Sec. 4) that $w\ltimes$ holds.

We show two more examples of programmer assertions. The first concerns *observational purity* [6]. Assume we repeatedly need to apply a function $expensive(z)$, the computation of which is very expensive. To save time, we decide to memoize the most recent call[7]. For that purpose, we introduce a class M, with fields $marg$ and res obeying the invariant

$$(marg \neq 0) \Rightarrow (res = expensive(marg))$$

and with a method

int $cexp($**int** $z)\{$
  if $z = self.marg$
  then $result := self.res$
  else //compute $expensive(z)$ and store the value in $result$
    $result := expensive(z);\ self.marg := z;\ self.res := result$
  assert $(result = expensive(z))\}$

Obviously, the last assertion should not be checked at runtime (this would defy the purpose of memoization), but might instead be verified by a theorem prover, using the above-mentioned invariant.

Suppose we know that for *cexp*: (a) its *result* depends *only* on z, not on memo data (*marg* or *res*) and (b) its *computation* affects *only* an abstract location L_1. If L_1 is not used elsewhere, we can consider calls to *cexp* "observationally pure" [6]; this notion of purity is under consideration for extending JML [11] which currently disallows effectful method calls in assertions.

It remains to show (a) and (b). Indeed, in Sec. 4.3, we will see that from $z\ltimes$ and the programmer assertion, $result = expensive(z)$, we can derive $result\ltimes$. Hence it is easy to see that if $self \rightsquigarrow L_1$ and $z\ltimes$ are preconditions for *cexp*, then $result\ltimes$ is a valid postcondition for *cexp*. We also observe that $L_1.marg$ and $L_1.res$ are the only abstract addresses that may be modified by *cexp*. This information appears in the following method summary for *cexp*:

$$\{self \rightsquigarrow L_1,\ z\ltimes\}\ \_\ \{result\ltimes\}\ [L_1.marg, L_1.res].$$

Our second example with programmer assertions deals with *selective dependency* and we consider an example due to Cohen [13]: the command $b := x + a \ \mathsf{mod}\ 4$ where, clearly, b is not independent of a. However, only the lower order two bits of a are revealed to b; nothing else is revealed. Suppose we fix the lower order two bits of a to 3, i.e., $a \ \mathsf{mod}\ 4 = 3$. Then we can prove that the "rest of a is protected from b", by means of the derivation[8]

$\{x\ltimes\}$
  assert $a \ \mathsf{mod}\ 4 = 3;$
$\{a \ \mathsf{mod}\ 4 = 3, x\ltimes\}$
$\{(a \ \mathsf{mod}\ 4)\ltimes,\ x\ltimes\}$ (by logical implication)
  $b := x + a \ \mathsf{mod}\ 4;$
$\{b\ltimes\}$

That is, $b\ltimes$ is in the postcondition, under the assumption that $x\ltimes$ is in the precondition, but *without* assuming that $a\ltimes$ is too.

[7] The generalization to full memoization appears in Sec. 6.

[8] The technical development in this paper does not allow assertions $E\ltimes$ with E an expression, but it is straightforward to add them.

$$
\begin{array}{lll}
T & ::= & \textbf{int} \mid C \quad \text{data type} \\
CL & ::= & \textbf{class } C \ \{ \ \overline{T}\,\overline{f}; \overline{M} \ \} \quad \text{class declaration} \\
M & ::= & T \ m(U \ u) \ \{S\} \quad \text{method declaration} \\
S & ::= & x := E \mid x.f := y \quad \text{assign to variable, to field} \\
 & \mid & x := \textbf{new } C \mid x := y.f \quad \text{object construction, field access} \\
 & \mid & x := y.m(z) \mid S \,; S \quad \text{method call, sequence} \\
 & \mid & \textbf{if } x \textbf{ then } S \textbf{ else } S \mid \textbf{while } x \textbf{ do } S \quad \text{conditional, while} \\
 & \mid & \textbf{assert } \theta \quad \text{programmer assertion} \\
E & ::= & x \mid c \mid \textbf{null} \mid E \ \mathsf{op} \ E \mid k(E) \\
 & & \text{variable, constants, arith. operations, arith. functions} \\
\theta & ::= & x = c \mid x = y \mid x = k(E) \mid \ldots \quad \text{primitive assertions} \\
 & \mid & x = y.f \mid \ldots \quad \text{assertions involving fields} \\
 & \mid & \theta \wedge \theta \mid \theta \vee \theta
\end{array}
$$

Figure 1: BNF of language

The Rest of the Paper

Sec. 3 formalizes the language. Sec. 4 gives the syntax and semantics of assertions. Sec. 5 specifies the logic. The full memoization example, illustrating reasoning in the logic, appears in Sec. 6. Sec. 7 is about the computing of assertions and strongest postcondition. Sec. 8 concludes.

3 Language: syntax and semantics

Syntax. Our core language (Figure 1) is a class-based object-oriented language with recursive classes, methods and field update. We shall consider subclassing and dynamic dispatch; it should be straightforward to add cast and type test, but we have not done that yet. The grammar is based on given sets of class names (with typical element C), expressions (E), constants ranging over integers (c), field names (f), and method names (m). The names x, y, z, w are used for program variables, and k is used for mathematical functions (e.g., mod).

The BNF is self-explanatory. One difference from usual security-typed languages is that programmer assertions are allowed via the command **assert** θ. Conjunctions and disjunctions of programmer assertions are also allowed. A type is either a base type **int**, or a "class type", i.e., a class name C; like Java, we have nominal (by name) typing. We assume a function, *type*, that assigns a type to all program variables and to all fields. We also assume the existence of a class table, CT, that maps a class name to the corresponding class declaration. A class declaration consists of a class name, e.g., C, together with a list of *public* field declarations, e.g., $\overline{T}\ \overline{f}$, and a list of method declarations, e.g., $\overline{M}$. Consider a method m declared as $T\ m(U\ u)\ \{S\}$ in class C; such a method has return type T, and formal parameter type U, and body S where S is a command. To simplify the presentation, we shall assume that if class C has a declaration of method m then also any subclass of C has an explicit declaration of m (with the same formal parameter); this is no restriction since it amounts to stating that each subclass should either override, or copy, the methods of its parent. We employ a distinguished variable *result* such that the effect of an explicit return expression, **return** E, can be achieved by letting the last assignment of S be *result* $:= E$.

Semantics. We specify the semantics in relational style; such a semantics fits well with a Hoare-style partial correctness specification and eases the proofs, especially since our analysis is termination insensitive. After a brief description of the semantic domains involved, we define the semantics of commands and finally the semantics of well formed class tables.

[Assert]
$$\begin{aligned}&(s_0,h_0)\,[\llbracket \mathbf{assert}\ \theta\rrbracket\mu]\,(s,h) \iff\\ &\quad \llbracket\theta\rrbracket(s_0,h_0) \wedge\ s = s_0 \wedge\ h = h_0\end{aligned}$$

[Assign]
$$\begin{aligned}&(s_0,h_0)\,[\llbracket x := E\rrbracket\mu]\,(s,h) \iff\\ &\quad (\exists v \cdot v = \llbracket E\rrbracket s_0 \wedge\ s = [s_0 \mid x \mapsto v]) \wedge\ h = h_0\end{aligned}$$

[FieldAcc]
$$\begin{aligned}&(s_0,h_0)\,[\llbracket x := y.f\rrbracket\mu]\,(s,h) \iff\\ &\quad \exists \ell \in Loc \cdot (s_0(y) = \ell\\ &\qquad \wedge\ s = [s_0 \mid x \mapsto h_0\,\ell\,f])\\ &\quad \wedge\ h = h_0\end{aligned}$$

[FieldUpd]
$$\begin{aligned}&(s_0,h_0)\,[\llbracket x.f := y\rrbracket\mu]\,(s,h) \iff\\ &\quad s = s_0 \wedge\ \exists \ell \in Loc \cdot (s_0(x) = \ell\\ &\qquad \wedge\ h = [h_0 \mid \ell.f \mapsto s_0(y)])\end{aligned}$$

[New]
$$\begin{aligned}&(s_0,h_0)\,[\llbracket x := \mathbf{new}\ C\rrbracket\mu]\,(s,h) \iff\\ &\quad \exists \ell \cdot (\mathit{type}\,\ell = C\ \wedge\ \ell \notin \mathit{rng}(s_0) \wedge\\ &\qquad \ell \notin \mathit{dom}(h_0)\ \wedge\ \ell \notin \mathit{rng}(h_0) \wedge\\ &\qquad s = [s_0 \mid x \mapsto \ell] \wedge\\ &\qquad h = [h_0 \mid \ell \mapsto \mathit{defaults}])\end{aligned}$$

[MethodCall]
$$\begin{aligned}&(s_0,h_0)\,[\llbracket x := y.m(z)\rrbracket\mu]\,(s,h) \iff\\ &\quad (s_0',h_0)\,\mu(C,m)\,(v,h) \wedge\ s = [s_0 \mid x \mapsto v]\\ &\text{where } \ell = s_0(y) \text{ and } C = \mathit{type}\,\ell\\ &\text{and } s_0' = [\mathit{pars}(m,C) \mapsto s_0(z), \mathit{self} \mapsto \ell]\end{aligned}$$

[Seq]
$$\begin{aligned}&(s_0,h_0)\,[\llbracket S_1\,;S_2\rrbracket\mu]\,(s,h) \iff\\ &\quad \exists (s_1,h_1) \cdot ((s_0,h_0)\,[\llbracket S_1\rrbracket\mu]\,(s_1,h_1)\\ &\qquad \wedge\ (s_1,h_1)\,[\llbracket S_2\rrbracket\mu]\,(s,h))\end{aligned}$$

[If]
$$\begin{aligned}&(s_0,h_0)\,[\llbracket \mathbf{if}\ x\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\rrbracket\mu]\,(s,h) \iff\\ &\quad (s_0(x) \in \mathsf{True} \wedge\ (s_0,h_0)\,[\llbracket S_1\rrbracket\mu]\,(s,h)) \vee\\ &\quad (s_0(x) \in \mathsf{False} \wedge\ (s_0,h_0)\,[\llbracket S_2\rrbracket\mu]\,(s,h))\end{aligned}$$

[While]
$$\begin{aligned}&(s_0,h_0)\,[\llbracket \mathbf{while}\ x\ \mathbf{do}\ S\rrbracket\mu]\,(s,h) \iff \exists n \cdot (s_0,h_0)\,R_n\,(s,h)\\ &\quad \text{where } R_n \text{ is given inductively by}\\ &\qquad \forall s,h: \quad (s,h)\,R_n\,(s,h) \quad \text{if } s(x) \in \mathsf{False}\\ &\qquad \forall s,h,s',h': \quad (s,h)\,R_n\,(s',h') \quad \text{if } s(x) \in \mathsf{True} \text{ and } (n > 0) \text{ and}\\ &\qquad\qquad \exists (s'',h'') \cdot ((s,h)\,[\llbracket S\rrbracket\mu]\,(s'',h'') \wedge\ (s'',h'')\,R_{n-1}\,(s',h'))\end{aligned}$$

Table 1: Semantics of commands

The state of a method in execution comprises a store, s, and a heap, h. A store s (in semantic domain *Store*) assigns values to local variables and parameters, where values are integer constants or locations or the distinguished entity *nil* (which is *not* a location). We use v to range over values, and assume that *Val*, the set of all values, is partitioned into two disjoint parts, True and False, where all locations belong to True. For locations, we assume given a countable set *Loc* ranged over by ℓ. We assume each location ℓ has a class C associated with it, and write $\mathit{type}\ \ell = C$. For all constants c we write $\mathit{type}\ c = \mathbf{int}$. For each type, we define a default value of that type: $\mathit{default}(\mathbf{int}) = 0$ and $\mathit{default}(C) = \mathit{nil}$.

A heap h (in semantic domain *Heap*) is a finite partial function from locations to object states, where an object state is a total mapping from field names to values. With abuse of notation, we say that location ℓ is in the range of heap h if there exists location ℓ_0 in $\mathit{dom}(h)$ and a field f such that $\ell = h\,\ell_0\,f$. We will work with *self-contained* states: say that state (s,h) is self-contained iff (a) for all ℓ in the range of s, ℓ is in the domain of h; and (b) for all ℓ in the range of h (c.f. above), ℓ is in the domain of h.

We shall consider only well-typed programs. In particular, we shall assume that if $s(x) = \ell$ ($h\,\ell_0\,f = \ell$) then $\mathit{type}\ \ell$ is a subclass of $\mathit{type}\ x$ ($\mathit{type}\ f$), and if $s(x)$ ($h\,\ell_0\,f$) is a constant c then $\mathit{type}\ x = \mathbf{int}$ ($\mathit{type}\ f = \mathbf{int}$).

The meaning, $\llbracket E\rrbracket$, of an expression, E, is a function from *Store* to *Val*; its definition is standard and thus elided. Pointer arithmetic is disallowed: in an expression $E = E_1\ \mathsf{op}\ E_2$, each $\llbracket E_i\rrbracket s$ has to evaluate to an integer and $\llbracket E\rrbracket s$ must be an integer; similarly for an expression $k(E)$. The meaning of an assertion θ is a predicate on states: $\llbracket\theta\rrbracket \in \mathit{Store} \times \mathit{Heap} \to \mathit{Bool}$.

The semantics of a class table is a method environment μ which provides a relational meaning, $\mu(C,m)$, for each method m declared in class C. The method environment μ is computed using a fixpoint construction. For each class C and method name m, $\mu(C,m) \subseteq (\mathit{Store} \times \mathit{Heap}) \times (\mathit{Val} \times \mathit{Heap})$.

Because a command S may contain method calls as constituents, the meaning of S is with respect to a method environment μ. More precisely, $\llbracket S\rrbracket\mu$ is a relation on input and output states: $\llbracket S\rrbracket\mu \subseteq (\mathit{Store} \times \mathit{Heap}) \times (\mathit{Store} \times \mathit{Heap})$. The relational semantics of commands appears in Table 1. We explain the cases [FieldUpd], [New] and [MethodCall] below.

In field update, $x.f := y$, the heap h_0 is updated with the value of y at field f of location ℓ, where ℓ is the meaning of x. (We use the notation $[h_0 \mid \ell.f \mapsto v]$ to denote the update of the object state $h_0\,\ell$ at field f by v).

In object allocation, $x :=$ **new** C, a fresh location ℓ of type C is allocated in the heap; the resulting store maps x to ℓ. The resulting heap, h, is the old heap, h_0, with its domain extended with ℓ. Each field f of C in the object state $h\,\ell$ is initialized to the default value of $type(f)$; this is captured by the notation $[h_0 \mid \ell \mapsto defaults]$.

For a method call, $x := y.m(z)$, suppose that y denotes a location ℓ with $type\,\ell = C$, where class C contains a method m with formal parameter u (written $pars(m, C) = u$). Let the initial state be (s_0, h_0), and suppose that the meaning of the method m is looked up in method environment μ, using a state whose heap component is h_0 but whose store component is a "local store", s_0', that binds *self* to ℓ and u to $s_0(z)$. Let the method meaning relate (s_0', h_0) to (v, h), where v is the return result of the method, and h the updated heap. Upon return, local store s_0' is discarded, and the resulting state is heap h together with the initial store, s_0, with x updated to v.

Observe that for some (s_0, h_0) there may be no (s, h) with $(s_0, h_0)\,[[\![S]\!]\mu]\,(s, h)$. This will be the case in the event of an infinite computation, a run-time error (like dereferencing a null pointer), or a failed programmer assertion.

We are now ready for the semantics of a class table, CT. The semantics makes explicit the fixpoint computation alluded to earlier.

Definition 3.1 (Semantics of class table, CT**)** *$[\![CT]\!]$ is the least upper bound (wrt. subset inclusion) of the ascending chain μ_n ($n \in \mathsf{Nats}$) of method environments, defined as follows (where class C contains method m with body S):*

$$
\begin{array}{l}
\mu_0(C, m) = \varnothing \\
(s_0, h_0)\,(\mu_{n+1}(C, m))\,(v, h) \iff \\
\qquad \exists s \cdot (s_0, h_0)\,[[\![S]\!]\mu_n]\,(s, h) \wedge (v = s(result))
\end{array}
$$

We now have the following technical results on the semantics of a class table, CT, where we let $\mu = [\![CT]\!]$.

Fact 3.2 *If $(s_0, h_0)\,(\mu(C, m))\,(v, h)$ then there exists n_0 such that for all $n \geq n_0$, $(s_0, h_0)\,(\mu_n(C, m))\,(v, h)$.*

Lemma 3.3 *If $(s_0, h_0)\,[[\![S]\!]\mu]\,(s, h)$ then there exists n_0 such that for all $n \geq n_0$, $(s_0, h_0)\,[[\![S]\!]\mu_n]\,(s, h)$.*

Proof: By induction in the derivation of $(s_0, h_0)\,[[\![S]\!]\mu]\,(s, h)$, using Fact 3.2. ■

Lemma 3.4 *Assume that $(s_0, h_0)\,[[\![S]\!]]\mu\,(s, h)$.*

(a) *If (s_0, h_0) is self-contained then so is (s, h).*

(b) $dom(s_0) \subseteq dom(s)$

(c) $dom(h_0) \subseteq dom(h)$

Proof: By Lemma 3.3, it is sufficient if for all n we can show that $(s_0, h_0)\,[[\![S]\!]\mu_n]\,(s, h)$ implies **(a)**, **(b)**, **(c)** above. We proceed by induction on n, with an inner induction on the derivation of $(s_0, h_0)\,[[\![S]\!]\mu_n]\,(s, h)$. An interesting case is for the [MethodCall] rule, where by the outer induction hypothesis we infer that if $(s_0, h_0)\,(\mu_n(C, m))\,(v, h)$ with (s_0, h_0) self-contained, then the state $([result \mapsto v], h)$ is self-contained and that $dom(h_0) \subseteq dom(h)$. ■

In the future, we shall implicitly assume that all states (s, h) in question are self-contained.

Creation and modification of state. Sec. 2 presented several examples of local reasoning that were justified by the frame rule. Such reasoning is sound because a side condition holds for the frame rule: when the small specification of a command is extended with other assertions, the abstract addresses mentioned in the assertions are *disjoint* from the corresponding abstract addresses *modified by* the command. Both notions are made precise in Sec. 4. But first Definition 3.5 states precisely what it means to modify concrete locations occurring in heaps and stores.

Definition 3.5 *For a location ℓ of type C, and for a field f of C, say that $\ell.f$ is* modified from heap h to heap h' *if $\ell \in dom(h')$ and either of the following conditions hold: (a) $\ell \in dom(h)$, and $h'\ell f \neq h\ell f$; (b) $\ell \notin dom(h)$, and $h'\ell f \neq default(type\ f)$.*

Variable x is modified from store s to store s' if $x \in dom(s')$ and either of the following conditions hold: (a) $x \in dom(s)$, and $s(x) \neq s'(x)$; (b) $x \notin dom(s)$.

Say that location ℓ is created from heap h to heap h' *provided $\ell \in dom(h')$ but $\ell \notin dom(h)$.*

Lemma 3.6 *Given s, s', s'' with $dom(s) \subseteq dom(s'') \subseteq dom(s')$, and h, h', h'' with $dom(h) \subseteq dom(h'') \subseteq dom(h')$.*

Assume that ℓ is created from h to h'. Then either ℓ is created from h to h'', or ℓ is created from h'' to h'.

Assume that x is modified from s to s'. Then either x is modified from s to s'', or x is modified from s'' to s'.

Assume that $\ell.f$ is modified from h to h'. Then either $\ell.f$ is modified from h to h'', or $\ell.f$ is modified from h'' to h'.

Proof: First assume that ℓ is created from h to h'. Then $\ell \in dom(h')$ but $\ell \notin dom(h)$. If $\ell \in dom(h'')$, then ℓ is created from h to h''; otherwise, ℓ is created from h'' to h'.

Next, assume that x is modified from s to s'. Then $x \in dom(s')$, and either:

1. $x \in dom(s)$, and $s'(x) \neq s(x)$. Then $x \in dom(s'')$, and either $s''(x) \neq s(x)$ or $s''(x) \neq s'(x)$, yielding the claim.

2. $x \notin dom(s)$. If $x \in dom(s'')$, then x is modified from s to s''; if $x \notin dom(s'')$, then x is modified from s'' to s'.

Finally, assume that $\ell.f$ is modified from h to h'. Then $\ell \in dom(h')$, and either:

1. $\ell \in dom(h)$, with $h'\,\ell\,f \neq h\,\ell\,f$. Then $\ell \in dom(h'')$. If $h'\,\ell\,f \neq h''\,\ell\,f$ then $\ell.f$ is modified from h'' to h'. Otherwise, $h''\,\ell\,f \neq h\,\ell\,f$, so $\ell.f$ is modified from h to h''.

2. $\ell \notin dom(h)$, and $h'\,\ell\,f \neq dv$ with $dv = default(type\ f)$. Then there are two possibilities:

   - $\ell \notin dom(h'')$, or $\ell \in dom(h'')$ with $h''\,\ell\,f = dv$. But then $\ell.f$ is modified from h'' to h'.
   - $\ell \in dom(h'')$ with $h''\,\ell\,f \neq dv$. But then $\ell.f$ is modified from h to h''.

■

4 Assertions

This section formalizes abstract locations, and provides the syntax and semantics of assertions. It also makes precise the two main ingredients of the frame rule alluded to in Sec. 2, namely, the modification of abstract addresses, and disjointness. The frame rule can only be applied when an assertion is disjoint from the set of abstract addresses that may be modified by a command.

Abstract Locations. We let L range over the set of abstract locations, *AbsLoc*. Think of L as a token that stands for a set of concrete heap locations. We will consider the following relations on *AbsLoc*: a partial ordering relation, $L_1 \preceq L_2$, conveys that L_2 contains at least those concrete heap locations that L_1 contains. We also need a symmetric relation, $L_1 \diamond L_2$, pronounced "L_1 is disjoint from L_2", to convey that L_1 and L_2 have no concrete heap locations in common. We add a special element $\perp$ to *AbsLoc* so that for all $L \in AbsLoc$ we have $\perp \preceq L$ and $\perp \diamond L$. One can think of $\perp$ as the counterpart of the concrete value *nil*.

We assume that if $L_1 \preceq L_2$ and $L \diamond L_2$ then also $L \diamond L_1$, and that if $L \neq \perp$ then $L \diamond L$ does not hold. We let LI range over $AbsLoc \cup \{\mathbf{int}\}$. An abstract entity is either an abstract address, x or $L.f$, or[9] an abstract location, L. We let X range over sets of abstract entities.

Syntax of assertions As noted in Sec. 2, we have three kinds of primitive assertions, namely, points-to assertions, agreement assertions, and programmer assertion. The BNF of assertions is this:

$$\begin{array}{lll}\phi & ::= & \theta \mid x \leadsto LI \mid L.f \leadsto LI \mid x\ltimes \mid L.f\ltimes \mid \mathit{true} \mid \phi \wedge \phi \\ & & \mid L\ \mathit{abs}\ i \quad (i = 0, 1) \\ & & \mid \phi\ \mathtt{uor}\ \phi \mid \phi\ \mathtt{ior}\ \phi\end{array}$$

An assertion is now formed by conjunction and/or disjunction of primitive assertions, Recall from Sec. 2 that we shall often use the set notation to denote conjunctions of assertions.

Roughly, the meaning of $x \leadsto L$ in a state (s, h) is that the concrete heap location denoted by x is abstracted by L. The meaning of $a\ltimes$ is that the *two* current states in question, say (s, h) and (s_1, h_1), agree on the value of a; agreement implies that there is no leak of information via a. This intuition leads to the one-state and two-state semantics for assertions in the sequel. The points-to assertion $L\ abs\ i$ states that L contains at most i concrete heap locations. We employ *two* kinds of disjunction[10]: the assertion ϕ_1 `ior` ϕ_2 states that either the two states in question jointly satisfy ϕ_1, or the two states in question jointly satisfy ϕ_2; the assertion ϕ_1 `uor` ϕ_2 states that each of the two states in question should satisfy either ϕ_1 or ϕ_2. We define functions $\mathcal{I}$ and $\mathcal{U}$ on assertions, with $\mathcal{I}$ factoring out the assertions related to information flow, and $\mathcal{U}$ factoring out the assertions unrelated to information flow.

$$\begin{array}{lll}\mathcal{U}(\theta) & = & \theta \\ \mathcal{U}(x \leadsto LI) & = & x \leadsto LI \\ \mathcal{U}(L.f \leadsto LI) & = & L.f \leadsto LI \\ \mathcal{U}(x\ltimes) = \mathcal{U}(L.f\ltimes) & = & \mathit{true} \\ \mathcal{U}(\mathit{true}) & = & \mathit{true} \\ \mathcal{U}(\phi_1 \wedge \phi_2) & = & \mathcal{U}(\phi_1) \wedge \mathcal{U}(\phi_2) \\ \mathcal{U}(L\ \mathit{abs}\ i) & = & L\ \mathit{abs}\ i \\ \mathcal{U}(\phi_1\ \mathtt{uor}\ \phi_2) & = & \mathcal{U}(\phi_1)\ \mathtt{uor}\ \mathcal{U}(\phi_2) \\ \mathcal{U}(\phi_1\ \mathtt{ior}\ \phi_2) & = & \mathcal{U}(\phi_1)\ \mathtt{ior}\ \mathcal{U}(\phi_2) \\ \\ \mathcal{I}(\theta) & = & \mathit{true} \\ \mathcal{I}(x\ltimes) & = & x\ltimes \\ \mathcal{I}(L.f\ltimes) & = & L.f\ltimes \\ \mathcal{I}(\mathit{true}) & = & \mathit{true} \\ \mathcal{I}(\phi_1 \wedge \phi_2) & = & \mathcal{I}(\phi_1) \wedge I(\phi_2) \\ \mathcal{I}(x \leadsto LI) & = & \mathit{true} \\ \mathcal{I}(L.f \leadsto LI) & = & \mathit{true} \\ \mathcal{I}(L\ \mathit{abs}\ i) & = & \mathit{true} \\ \mathcal{I}(\phi_1\ \mathtt{uor}\ \phi_2) & = & \mathit{true} \\ \mathcal{I}(\phi_1\ \mathtt{ior}\ \phi_2) & = & \mathcal{I}(\phi_1)\ \mathtt{ior}\ \mathcal{I}(\phi_2)\end{array}$$

[9] In order to accommodate assertions $L\ abs\ i$, enabling "strong update" but omitted in the short version of this paper.

[10] For practical applications, it might suffice to have only one, and one might perhaps even dispense with disjunction altogether.

One-state Semantics of Assertions. To give a precise meaning to assertions, we need to assume the existence of an extraction relation, η, (similar to the extraction functions described in [19, p.235]) that relates locations to abstract locations. We require that η satisfies the following properties:

- For all L_2, $\ell\,\eta\,L_2$ holds iff there exists L_1 such that $L_1 \preceq L_2$ and $\ell\,\eta\,L_1$.
- If $L_1 \diamond L_2$ then for no ℓ we have $\ell\,\eta\,L_1$ and $\ell\,\eta\,L_2$.
- $\ell\,\eta\,\bot$ holds for no ℓ.

For convenience, we extend η to Val, so that $c\,\eta\,\mathbf{int}$ and $nil\,\eta\,\bot$ – thus $nil\,\eta\,L$ holds for all L. But $c\,\eta\,L$ holds for no L, and $\ell\,\eta\,\mathbf{int}$ holds for no ℓ, and $nil\,\eta\,\mathbf{int}$ does not hold.

We say that η *is over* h if $\ell\,\eta\,L$ implies $\ell \in dom(h)$. For η over h, we are now in a position to define the semantics of an assertion ϕ in state (s,h), written, $(s,h) \models_\eta \phi$.

$$\begin{array}{lcl}
(s,h) \models_\eta \theta & \iff & [\![\theta]\!](s,h) \\
(s,h) \models_\eta x \rightsquigarrow LI & \iff & s(x)\,\eta\,LI \\
(s,h) \models_\eta L.f \rightsquigarrow LI & \iff & \forall \ell \in dom(h) \cdot \ell\,\eta\,L \Rightarrow (h\ell f)\,\eta\,(LI) \\
(s,h) \models_\eta x \ltimes & \iff & true \\
(s,h) \models_\eta L.f \ltimes & \iff & true \\
(s,h) \models_\eta true & \iff & true \\
(s,h) \models_\eta \phi_1 \wedge \phi_2 & \iff & (s,h) \models_\eta \phi_1 \text{ and } (s,h) \models_\eta \phi_2 \\
(s,h) \models_\eta L\ abs\ i & \iff & i \geq card(\{\ell \in dom(h) \mid \ell\,\eta\,L\}) \\
(s,h) \models_\eta \phi_1\ \mathtt{uor}\ \phi_2 & \iff & (s,h) \models_\eta \phi_1 \text{ or } (s,h) \models_\eta \phi_2 \\
(s,h) \models_\eta \phi_1\ \mathtt{ior}\ \phi_2 & \iff & (s,h) \models_\eta \phi_1 \text{ or } (s,h) \models_\eta \phi_2
\end{array}$$

A simple induction gives

Fact 4.1 $(s,h) \models_\eta \phi$ *iff* $(s,h) \models_\eta \mathcal{U}(\phi)$.

Two-state Semantics of Assertions. Consider, e.g., the assertion $x\ltimes$ and consider two states (s,h) and (s',h') for which we want the values of x to agree. If x denotes a location then, because of different allocation behavior in h and h', we cannot expect $s(x)$ and $s'(x)$ to be equal. Rather we expect the former to yield location ℓ and the latter to yield location ℓ', so that the agreement can be enforced by a bijection β that relates ℓ and ℓ'. On the other hand, not all locations need to be related to some other location, similar to what is the case for type-based information flow analysis [5]. There, the indistinguishability relation on states (s,h) and (s',h') is formalized using a bijection between those locations in $dom(h)$ and $dom(h')$ that are visible to a "low observer".

We formalize the above intuition. Let β range over bijections from a subset of Loc to a subset of Loc. That is, if $\ell\,\beta\,\ell_1$ and $\ell\,\beta\,\ell_2$ then $\ell_1 = \ell_2$, but for some ℓ_0 there might not be any ℓ' such that $\ell_0\,\beta\,\ell'$; and if $\ell_1\,\beta\,\ell$ and $\ell_2\,\beta\,\ell$ then $\ell_1 = \ell_2$, but for some ℓ_0 there might not be any ℓ' such that $\ell'\,\beta\,\ell_0$. In addition, with abuse of notation, for all integer constants c we shall assume that $c\,\beta\,c$, and also assume that $nil\,\beta\,nil$. Note that if $v\,\beta\,v_1$ then $v \in \mathsf{True}$ iff $v_1 \in True$ (since all locations belong to True). We say that β *is over* $h\&h_1$ if $\ell\,\beta\,\ell_1$ implies $\ell \in dom(h)$ and $\ell_1 \in dom(h_1)$.

We can now define the two-state semantics of assertion ϕ, written $(s,h)\&(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi$. Here β is over $h\&h_1$, and η is over h, and η_1 is over h_1; further, if $\ell\,\beta\,\ell_1$ then $type\,\ell = type\,\ell_1$, and $\ell\,\eta\,L$ iff $\ell_1\,\eta_1\,L$. The last condition simply says that concrete locations ℓ and ℓ_1 related by β are abstracted to the *same* abstract location L by both η and η_1.

$$(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ x\ltimes \iff (s\,x)\,\beta\,(s_1\,x)$$

$$\begin{array}{l}(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ L.f\ltimes \iff \\ \quad \forall \ell \in \mathit{dom}(h),\ \ell_1 \in \mathit{dom}(h_1)\cdot \\ \quad \ell\,\beta\,\ell_1 \wedge \ell\,\eta\,L \Rightarrow (h\,\ell\,f)\,\beta\,(h_1\,\ell_1\,f)\end{array}$$

$$\begin{array}{l}(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi \iff \\ \quad (s,h)\ \models_\eta\ \phi \text{ and } (s_1,h_1)\ \models_{\eta_1}\ \phi, (\phi \text{ is } x \leadsto L, L.f \leadsto LI, \theta, \mathit{true}, L\ \mathit{abs}\ i, \phi_1\ \mathtt{uor}\ \phi_2)\end{array}$$

$$\begin{array}{l}(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_1\wedge\phi_2 \iff \\ \quad (s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_1 \text{ and } (s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_2\end{array}$$

$$\begin{array}{l}(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_1\ \mathtt{ior}\ \phi_2 \iff \\ \qquad (s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_1 \text{ or } (s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_2\end{array}$$

4.1 Properties of Assertions

Lemma 4.2 *Assume that with $x_1,\ldots,x_n$ the only program variables in ϕ, ϕ' is given by in ϕ replacing $x_1,\ldots x_n$ by $y_1,\ldots,y_n$. Assume that s, s_1, s', s_1' are such that for all $i = 1\ldots n$, $s(x_i) = s'(y_i)$ and $s_1(x_i) = s_1'(y_i)$. Then $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi$ iff $(s',h)\,\&\,(s_1',h_1)\ \models_{\beta,\eta,\eta_1}\ \phi'$.*

Proof: An easy induction in ϕ, using a similar result for $\models_\eta$. ■

Lemma 4.3 *If $(s,h)\ \models_\eta\ \phi$, then, with β the identity on $\mathit{dom}(h)$, we have $(s,h)\,\&\,(s,h)\ \models_{\beta,\eta,\eta}\ \phi$.*

Proof: Go by structural induction on ϕ.

$\phi = x\ltimes$. Because (s,h) is self-contained, if $s(x)$ is a location then $s(x)\in \mathit{dom}(h)$, so in all cases we have the desired relation $(s\,x)\,\beta\,(s\,x)$.

$\phi = L.f\ltimes$. Consider $\ell,\ell_1 \in \mathit{dom}(h)$ with $\ell\,\beta\,\ell_1$. Then $\ell = \ell_1$, so $h\ell f = h\ell_1 f$, and thus (again since (s,h) is self-contained), $(h\ell f)\,\beta\,(h\ell_1 f)$.

$\phi = x \leadsto LI$ or $\phi = L.f \leadsto LI$ or $\phi = \theta$ or $\phi = \mathit{true}$ or $\phi = L\ \mathit{abs}\ i$ or $\phi = \phi_1\ \mathtt{uor}\ \phi_2$. All follow by definition.

$\phi = \phi_1\wedge\phi_2$. Follows easily using induction hypothesis.

$\phi = \phi_1\ \mathtt{ior}\ \phi_2$. Wlog, we can assume $(s,h)\ \models_\eta\ \phi_1$. Inductively, $(s,h)\,\&\,(s,h)\ \models_{\beta,\eta,\eta}\ \phi_1$ and therefore also $(s,h)\,\&\,(s,h)\ \models_{\beta,\eta,\eta}\ \phi$. ■

Lemma 4.4 *Assume $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi$.*
Then $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \mathcal{I}(\phi)$ and $(s,h)\ \models_\eta\ \mathcal{U}(\phi)$ and $(s_1,h_1)\ \models_{\eta_1}\ \mathcal{U}(\phi)$.

Note that by Fact 4.1, the conclusion entails $(s,h)\ \models_\eta\ \phi$ and $(s_1,h_1)\ \models_{\eta_1}\ \phi$.

Proof: Go by structural induction on ϕ.

$\phi = x\ltimes$ or $L.f\ltimes$. Then $\mathcal{I}(\phi) = \phi$ and $\mathcal{U}(\phi) = \mathit{true}$, so the claim is trivial.

$\phi = x \leadsto L$ or $\phi = L.f \leadsto LI$ or or $\phi = \theta$ or $\phi = \mathit{true}$ or $\phi = L\ \mathit{abs}\ i$. Then $\mathcal{I}(\phi) = \mathit{true}$ and $\mathcal{U}(\phi) = \phi$, and the claim follows by the definition of $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi$.

$\phi = \phi_1\wedge\phi_2$. We have $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_1$ and $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_2$. So inductively: $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \mathcal{I}(\phi_1)$ and $(s,h)\ \models_\eta\ \mathcal{U}(\phi_1)$ and $(s_1,h_1)\ \models_{\eta_1}\ \mathcal{U}(\phi_1)$ and $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \mathcal{I}(\phi_2)$ and $(s,h)\ \models_\eta\ \mathcal{U}(\phi_2)$ and $(s_1,h_1)\ \models_{\eta_1}\ \mathcal{U}(\phi_2)$. Thus (by reordering), $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \mathcal{I}(\phi)$ and $(s,h)\ \models_\eta\ \mathcal{U}(\phi)$ and $(s_1,h_1)\ \models_{\eta_1}\ \mathcal{U}(\phi)$.

$\underline{\phi = \phi_1 \texttt{ ior } \phi_2.}$ Wlog, we can assume that $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi_1$. Inductively, $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \mathcal{I}(\phi_1)$ and $(s,h) \models_\eta \mathcal{U}(\phi_1)$ and $(s_1,h_1) \models_{\eta_1} \mathcal{U}(\phi_1)$ implying $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \mathcal{I}(\phi_1) \texttt{ ior } \mathcal{I}(\phi_2)$ and $(s,h) \models_\eta \mathcal{U}(\phi_1) \texttt{ ior } \mathcal{U}(\phi_2)$ and $(s_1,h_1) \models_{\eta_1} \mathcal{U}(\phi_1) \texttt{ ior } \mathcal{U}(\phi_2)$ which amounts to the desired relation.

$\underline{\phi = \phi_1 \texttt{ uor } \phi_2.}$ Then $(s,h) \models_\eta \phi_1 \texttt{ uor } \phi_2$ and $(s_1,h_1) \models_{\eta_1} \phi_1 \texttt{ uor } \phi_2$ and by Fact 4.1, therefore $(s,h) \models_\eta \mathcal{U}(\phi_1 \texttt{ uor } \phi_2)$ and $(s_1,h_1) \models_{\eta_1} \mathcal{U}(\phi_1 \texttt{ uor } \phi_2)$. This is as desired, since $\mathcal{I}(\phi_1 \texttt{ uor } \phi_2) = \mathit{true}$. ■

The converse of Lemma 4.4 does not hold if ϕ contains `ior`: to see that, let $\phi = x \rightsquigarrow L \texttt{ ior } x \rightsquigarrow L_1$ and $s(x) = \ell$ and $s_1(x) = \ell_1$, with $\ell\,\eta\,L$ and with $\ell_1\,\eta\,L_1$ but not $\ell\,\eta\,L_1$ and not $\ell_1\,\eta\,L$. Then $\mathcal{I}(\phi) = \mathit{true} \texttt{ ior } \mathit{true}$ and $\mathcal{U}(\phi) = x \rightsquigarrow L \texttt{ ior } x \rightsquigarrow L_1$. So $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta} \mathcal{I}(\phi)$ and $(s,h) \models_\eta \mathcal{U}(\phi)$ and $(s_1,h_1) \models_\eta \mathcal{U}(\phi)$ but $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta} \phi$ does not hold since we have neither $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta} x \rightsquigarrow L$ (as $s_1(x)\,\eta\,L$ does not hold) nor $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta} x \rightsquigarrow L_1$.

Lemma 4.5 *Assume that ϕ does not contain* `ior`*.*
If $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \mathcal{I}(\phi)$ and $(s,h) \models_\eta \mathcal{U}(\phi)$ and $(s_1,h_1) \models_{\eta_1} \mathcal{U}(\phi)$ then $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi$.

Proof: Go by structural induction on ϕ.

$\underline{\phi = x \ltimes \text{ or } L.f \ltimes.}$ Then $\mathcal{I}(\phi) = \phi$, so the claim is trivial.

$\underline{\phi = x \rightsquigarrow L \text{ or } \phi = L.f \rightsquigarrow LI \text{ or } \phi = \theta \text{ or } \phi = \mathit{true} \text{ or } \phi = L \; \mathit{abs} \; i.}$ Then $\mathcal{U}(\phi) = \phi$, and the claim follows by the definition of $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi$.

$\underline{\phi = \phi_1 \wedge \phi_2.}$ Assume $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \mathcal{I}(\phi)$ and $(s,h) \models_\eta \mathcal{U}(\phi)$ and $(s_1,h_1) \models_{\eta_1} \mathcal{U}(\phi)$.
Then $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \mathcal{I}(\phi_1)$ and $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \mathcal{I}(\phi_2)$ and $(s,h) \models_\eta \mathcal{U}(\phi_1)$ and $(s,h) \models_\eta \mathcal{U}(\phi_2)$ and $(s_1,h_1) \models_{\eta_1} \mathcal{U}(\phi_1)$ and $(s_1,h_1) \models_{\eta_1} \mathcal{U}(\phi_2)$. So inductively we have $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi_1$ and $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi_2$ and therefore the desired $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi$.

$\underline{\phi = \phi_1 \texttt{ uor } \phi_2.}$ Assume $(s,h) \models_\eta \mathcal{U}(\phi)$ and $(s_1,h_1) \models_{\eta_1} \mathcal{U}(\phi)$. By Fact 4.1, $(s,h) \models_\eta \phi$ and $(s_1,h_1) \models_{\eta_1} \phi$ and therefore the desired $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi$.
■

4.2 Dynamic Properties of Assertions

We now specify the conditions under which a set of abstract entities X are created/modified from state (s,h) to state (s',h') under extraction relation η over heap h'. This is written, $(s,h) \rightarrow (s',h') \models_\eta X$. The set of abstract entities in X overapproximates the set of concrete locations that may be created/modified from (s,h) to (s',h').

Definition 4.6 (Creating/Modifying an abstract entity)
Say that $(s,h) \rightarrow (s',h') \models_\eta X$ iff

(a) *for all y modified from s to s', $y \in X$.*

(b) *for all $\ell.f$ modified from h to h', there exists L with $\ell\,\eta\,L$ such that $L.f \in X$.*

(c) *for all ℓ created from h to h', there exists $L \in X$ with $\ell\,\eta\,L$.*

Disjointness. Recall that $L_1 \diamond L_2$ denotes that L_1 and L_2 are disjoint. We extend disjointness in two stages. In the first stage, we lift $\diamond$ to a relation between an abstract entity and a set of abstract entities as follows: (a) $x \diamond X$ iff $x \notin X$; (b) $L.f \diamond X$ iff for all $L_1.f \in X$, we have $L \diamond L_1$; (c) $L \diamond X$ iff for all $L_1 \in X$, we have $L \diamond L_1$ [11].

Second, we define what it means for an assertion ϕ to be disjoint from a set of abstract entities, X. This relation, written $\phi \diamond X$, holds provided $a \diamond X$ for all abstract addresses a occurring on "the left hand side" of assertions in ϕ, and $L \diamond X$ for all $L \; \mathit{abs} \; i$ in ϕ:

[11] Note that it is possible for $L \diamond X$ to hold even if X contains $L.f$.

Definition 4.7 *We write $\phi \diamond X$ when all of the following holds:*

- *For all $x \rightsquigarrow LI$ occurring in ϕ, $x \diamond X$.*
- *For all $L.f \rightsquigarrow LI$ occurring in ϕ, $L.f \diamond X$.*
- *For all L abs i occurring in ϕ, $L \diamond X$.*
- *For all $x\ltimes$ occurring in ϕ, $x \diamond X$.*
- *For all $L.f\ltimes$ occurring in ϕ, $L.f \diamond X$.*
- *For all θ occurring in ϕ: if x occurs in θ then $x \notin X$; if $x.f$ occurs in θ then*[12] *no $L.f$ occurs in X.*

As we shall see later (Sec. 5), $\phi \diamond X$ is exactly the form of the side condition of the frame rule.

The main result of this section is an invariance result, intuitively stating that an assertion which is valid *before* executing a command, also remains valid *after*, provided it is *disjoint* from any abstract entity modified/created by the command.

To precisely state this result, we need the following notion of "extension" of η and β: Say that η' over h' *extends* η over h, if $dom(h) \subseteq dom(h')$ and for all $\ell \in dom(h)$, for all L: $\ell\,\eta\,L$ iff $\ell\,\eta'\,L$.

Let $dom(h) \subseteq dom(h')$ and $dom(h_1) \subseteq dom(h_1')$. Say that β' over $h'\&h_1'$ *extends* β over $h\&h_1$ if $\beta = \{(\ell,\ell_1) \in \beta' \mid (\ell \in dom(h)) \vee (\ell_1 \in dom(h_1))\}$. (Therefore, if $\ell\,\beta'\,\ell_1$ and $\ell \in dom(h)$ then $\ell_1 \in dom(h_1)$, and vice versa).

Lemma 4.8 *Assume that η' over h' extends η'' over h'', and that η'' over h'' extends η over h. Then η' over h' extends η over h.*

Proof: For all $\ell \in dom(h)$, all L: $\ell\,\eta\,L$ iff $\ell\,\eta''\,L$ iff $\ell\,\eta'\,L$. ■

Lemma 4.9 *Assume that β' over $h'\&h_1'$ extends β'' over $h''\&h_1''$, and that β'' over $h''\&h_1''$ extends β over $h\&h_1$. Then β' over $h'\&h_1'$ extends β over $h\&h_1$.*

Proof: We must show that $\beta = \{(\ell,\ell_1) \in \beta' \mid \ell \in dom(h) \text{ or } \ell_1 \in dom(h_1)\}$. So first consider $(\ell,\ell_1) \in \beta$. Our assumptions entail first $(\ell,\ell_1) \in \beta''$ and then $(\ell,\ell_1) \in \beta'$. Conversely, given $(\ell,\ell_1) \in \beta'$ with $\ell \in dom(h)$ or $\ell_1 \in dom(h_1)$. Then also $\ell \in dom(h'')$ or $\ell_1 \in dom(h_1'')$ which since β' extends β'' implies that $(\ell,\ell_1) \in \beta''$. But since β'' extends β, this implies the desired $(\ell,\ell_1) \in \beta$. ■

Lemma 4.10 *Assume that $(s,h) \to (s'',h'') \models_{\eta''} X_1$ and $(s'',h'') \to (s',h') \models_{\eta'} X_2$ where η' over h' extends η'' over h''. Then also $(s,h) \to (s',h') \models_{\eta'} X_1 \cup X_2$.*

Proof: First assume that x is modified from s to s'. By Lemma 3.6, either x is modified from s to s'' or x is modified from s'' to s'; our assumptions then imply that $x \in X_1$ or $x \in X_2$.

Next, assume that ℓ is created from h to h'. By Lemma 3.6, there are two cases: (a) ℓ is created from h to h'': By our assumption, there exists L with $\ell\,\eta''\,L$ such that $L \in X_1$ and thus $\ell\,\eta'\,L$ with $L \in X_1 \cup X_2$. (b) ℓ is created from h'' to h': By our assumption, there exists L with $\ell\,\eta'\,L$ such that $L \in X_2$, and thus $\ell\,\eta'\,L$ with $L \in X_1 \cup X_2$.

Finally, assume that $\ell.f$ is modified from h to h'. By Lemma 3.6, we have two cases: (a) $\ell.f$ is modified from h to h'': By our assumption, there exists L with $\ell\,\eta''\,L$ such that $L.f \in X_1$, and thus $\ell\,\eta'\,L$ with $L.f \in X_1 \cup X_2$. (b) $\ell.f$ is modified from h'' to h'. By our assumption, there exists L with $\ell\,\eta'\,L$ such that $L.f \in X_2$, and thus $\ell\,\eta'\,L$ with $L.f \in X_1 \cup X_2$. ■

Lemma 4.11 (Invariance) *Suppose $\phi \diamond X$. Further, suppose $(s,h) \to (s',h') \models_{\eta'} X$, and $(s_1,h_1) \to (s_1',h_1') \models_{\eta_1'} X$, where η' over h' extends η over h, and η_1' over h_1' extends η_1 over h_1. Also, let β' over $h'\&h_1'$ extend β over $h\&h_1$. Suppose $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi$. Then*

[12] This is a very crude requirement; we might be able to settle for something more precise like: *no $L.f$ occurs in X where ϕ "associates" x with L.*

$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta',\eta',\eta'_1} \phi$.

Proof: Go by structural induction on ϕ. The inductive case for $\wedge$ is trivial. Similarly, the inductive case for `ior` is trivial.

For the case where $\phi = \phi_1$ `uor` ϕ_2, our assumptions entail that

$$(s, h) \models_\eta \phi \tag{1}$$

$$(s_1, h_1) \models_{\eta_1} \phi \tag{2}$$

From (1), we can assume wlog. that $(s, h) \models_\eta \phi_1$. By Lemma 4.3 with ι the identity on $dom(h)$, $(s, h) \,\&\, (s, h) \models_{\iota,\eta,\eta} \phi_1$. We can conclude inductively that $(s', h') \,\&\, (s', h') \models_{\iota',\eta',\eta'} \phi_1$, with ι' the identity on $dom(h')$, and by Lemma 4.4 therefore $(s', h') \models_{\eta'} \phi_1$ from which we conclude $(s', h') \models_{\eta'} \phi$. Similarly, from (2) we can conclude $(s'_1, h'_1) \models_{\eta'_1} \phi$. But this shows the desired $(s', h') \,\&\, (s'_1, h'_1) \models_{\beta',\eta',\eta'_1} \phi$.

Next consider the base cases, where $\phi = true$ is trivial. For the base cases $y \rightsquigarrow LI$ and $L.f \rightsquigarrow LI$ and $L\ abs\ i$ and θ, for reasons of symmetry it suffices to prove $(s', h') \models_{\eta'} \phi$; for that purpose we can assume that $(s, h) \models_\eta \phi$.

ϕ is of the form θ. So $[\![\theta]\!](s, h)$ holds. From $\phi \diamond \theta$ we know that for all x occurring free in θ we have $x \notin X$, so $s'(x) = s(x)$. Now assume that also $x.f$ occurs in θ, with $s(x) = s'(x) = \ell$ where $\ell \in dom(h)$ and $\ell \in dom(h')$. We must prove that $h\ell f = h'\ell f$. Assume otherwise: then $\ell.f$ is modified from h to h' so by our assumption, there exists L with $\ell\, \eta'\, L$ such that $L.f \in X$. But by definition of $\phi \diamond X$, no $L.f$ belongs to X, yielding the desired contradiction. By collecting the pieces, we get the desired $[\![\theta]\!](s', h')$.

ϕ is of the form $L\ abs\ i$. So $i \geq card(\{\ell \in dom(h) \mid \ell\, \eta\, L\})$. We must prove that $i \geq card(\{\ell \in dom(h') \mid \ell\, \eta'\, L\})$ which can be done by establishing that

$$\{\ell \in dom(h) \mid \ell\, \eta\, L\} = \{\ell \in dom(h') \mid \ell\, \eta'\, L\}.$$

Here, $\subseteq$ is obvious. To prove $\supseteq$, assume that $\ell \in dom(h')$ with $\ell\, \eta'\, L$. If $\ell \in dom(h)$, then $\ell\, \eta\, L$ follows (since η' extends η). If $\ell \notin dom(h)$, then ℓ is created from h to h', so there exists $L_1 \in X$ with $\ell\, \eta'\, L_1$. But then we do not have $L \diamond L_1$, contradicting $\phi \diamond X$.

ϕ is of the form $y \rightsquigarrow LI$. So $s(y)\, \eta\, LI$. From $\phi \diamond X$, we infer that $y \notin X$. But then $s'(y) = s(y)$, so we have the desired $s'(y)\, \eta'\, LI$.

ϕ is of the form $L.f \rightsquigarrow LI$. We are given $\ell \in dom(h')$ with $\ell\, \eta'\, L$, and must prove that $(h'\ell f)\, \eta'\, LI$. We observe that $\ell.f$ is not modified from h to h'. For assume otherwise: then $\ell\, \eta'\, L_1$ with $L_1.f \in X$, so from $\phi \diamond X$ we infer that $L \diamond L_1$, contradicting $\ell\, \eta'\, L$ and $\ell\, \eta'\, L_1$. We consider two cases: (a) $\ell \in dom(h)$: then $h\ell f = h'\ell f$. Also, $\ell\, \eta\, L$, so from $(s, h) \models_\eta \phi$ we infer that $(h\ell f)\, \eta\, LI$. But then also $(h'\ell f)\, \eta'\, LI$, as desired. (b) $\ell \notin dom(h)$: then the value of $h'\ell f$ is the default value for the type of f. If $type\, f$ is **int**, then $LI = \mathbf{int}$ so the claim follows since $0\, \eta'\, \mathbf{int}$. Otherwise, $LI \neq \mathbf{int}$ and the claim follows since then $nil\, \eta'\, LI$.

ϕ is of the form $y \ltimes$. So $(sy)\, \beta\, (s_1 y)$. Here, $y \notin X$ (since $\phi \diamond X$). Therefore, $s'(y) = s(y)$ and $s'_1(y) = s_1(y)$. But this implies the desired $(s'y)\, \beta'\, (s'_1 y)$.

ϕ is of the form $L.f \ltimes$. Let $\ell \in dom(h')$ and $\ell_1 \in dom(h'_1)$ with $\ell\, \beta'\, \ell_1$ and $\ell\, \eta'\, L$ (thus also $\ell_1\, \eta'_1\, L$). We must prove that $(h'\ell f)\, \beta'\, (h'_1 \ell_1 f)$. From $\phi \diamond X$ we infer, as in a previous case, that $\ell.f$ is not modified from h to h', nor is $\ell.f$ modified from h_1 to h'_1. Since β' extends β, there are two possibilities: (a) $\ell \in dom(h), \ell_1 \in dom(h_1)$. Then $\ell\, \beta\, \ell_1$, $\ell\, \eta\, L$, $\ell_1\, \eta_1\, L$, $h\ell f = h'\ell f$, $h_1 \ell_1 f = h'_1 \ell_1 f$. We infer from our assumptions that $(h\ell f)\, \beta\, (h_1 \ell_1 f)$ and thus $(h'\ell f)\, \beta\, (h'_1 \ell_1 f)$ implying the desired $(h'\ell f)\, \beta'\, (h'_1 \ell_1 f)$. (b) $\ell \notin dom(h)$ and $\ell_1 \notin dom(h_1)$. Then the value of $h'\ell f$ is the default for the type of f, and similarly for $h'_1 \ell_1 f$. Then $(h'\ell f)\, \beta'\, (h'_1 \ell_1 f)$ trivially holds.

■

4.3 Logical implication

The purpose of this section is to define a notion of implication of assertions; this permits the deduction of more agreement assertions than can be obtained by tracking data and control flow only.

Definition 4.12 (Logically implies) *Say that ϕ_0 logically implies ϕ, written $\phi_0 \blacktriangleright \phi$, iff $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi_0$ implies $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \phi$.*

Note that if $L_1 \preceq L_2$ then we have the following logical implications:

- $x \rightsquigarrow L_1$ logically implies $x \rightsquigarrow L_2$.
- $L.f \rightsquigarrow L_1$ logically implies $L.f \rightsquigarrow L_2$.
- $L_2.f \rightsquigarrow LI$ logically implies $L_1.f \rightsquigarrow LI$.
- $L_2\ abs\ i$ logically implies $L_1\ abs\ i$.
- $L_2.f \ltimes$ logically implies $L_1.f \ltimes$.

The above definition allows us to show that the following logical implications are valid.

- Let θ be the programmer assertion $x = c$. Then $\theta \blacktriangleright x\ltimes$.
  For assume that $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \theta$. Then by semantics of agreement assertions, $(s,h) \models_\eta \theta$ and $(s_1,h_1) \models_{\eta_1} \theta$. That is, $s(x) = c$ and $s_1(x) = c$. But then $(s\ x)\,\beta\,(s_1\ x)$.
- Let θ be the assertion $(x = y)$. Then $(\theta \wedge y\ltimes) \blacktriangleright x\ltimes$. For assume that $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \theta \wedge y\ltimes$. Then by semantics of agreement assertions, $(s,h) \models_\eta \theta$ and $(s_1,h_1) \models_{\eta_1} \theta$ and $(s\ y)\,\beta\,(s_1\ y)$. Thus we get $s(x) = s(y)$ and $s_1(x) = s_1(y)$, implying $(s\ x)\,\beta\,(s_1\ x)$. Hence $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} x\ltimes$.
- Let θ be the assertion $x = k(y)$, with k an arithmetic function. Then $(\theta \wedge y\ltimes) \blacktriangleright x\ltimes$. For assume that $(s,h)\,\&\,(s_1,h_1) \models_{\beta,\eta,\eta_1} \theta \wedge y\ltimes$. Then $(s\ y)\,\beta\,(s_1\ y)$, and since $s\ y$ and $s_1\ y$ are integers this amounts to $s\ y = s_1\ y$ and therefore $k(s\ y) = k(s_1\ y)$. From $(s,h) \models_\eta \theta$ we infer $s(x) = k(s\ y)$; similarly we infer $s_1(x) = k(s_1\ y)$. We conclude $s(x) = s_1(x)$ which amounts to the desired $(s\ x)\,\beta\,(s_1\ x)$.

Several other such logical implications are possible. For applications, recall Sec. 2, and see Sec. 6.

We can define $\blacktriangleright$ on sets of abstract entities in a manner similar to Def. 4.12. Say that $X \blacktriangleright X'$ iff $(s,h) \rightarrow (s',h') \models_\eta X$ implies $(s,h) \rightarrow (s',h') \models_\eta X'$.

Fact 4.13 *If*

- *$x \in X$ implies $x \in X'$*
- *$L \in X$ implies there exists $L' \in X'$ with $L \preceq L'$*
- *$L.f \in X$ implies there exists L' with $L \preceq L'$ such that $L'.f \in X'$*

then $X \blacktriangleright X'$. In particular, if $X \subseteq X'$ then $X \blacktriangleright X'$.

5 Statically Checking Assertions via a Logic

To statically check assertions we define, in Table 2, a Hoare-like logic whose judgements take the form

$$\Pi \vdash \{\phi_0\}\ S\ \{\phi\}\ [X].$$

In the judgement, X is a set of abstract entities that overapproximates the abstract entities modified/created by S, ϕ_0 are the assertions that hold before execution of S, and ϕ are the assertions that hold after execution of S. Π is a summary environment for methods, such that $\Pi(C, m)$ is a (set of) summaries of the form $\{\psi_0\}$ _ $\{\psi\}$ $[X']$, where the only program variables mentioned in ψ_0 are *self* and the formal parameter of m, where the only program variable mentioned in ψ is *result*, and where X' does not contain program variables. The reason for having a *set* of summaries is polyvariance: at different call sites of the same method, different pre-and postconditions may hold. (This is similar to intersection type systems, where, given that variable x has an intersection type, a particular use of x selects the appropriate conjunct.) We will often omit Π in rules other than the rule for method call. Each judgement in Table 2 is a small specification.

Before discussing the small specifications in more detail, we shall define, for a judgement $\{\phi_0\}\ S\ \{\phi\}\ [X]$, its intended *meaning*, of which our logic will be a sound (but necessarily not complete) approximation.

5.1 Semantics of Judgements

Definition 5.1 *We say that* $\mu \models \{\phi_0\}\ S\ \{\phi\}\ [X]$ *iff the following holds for all* $s, h, s', h', s_1, h_1, s_1', h_1', \beta, \eta, \eta_1$. *Assume*

$$(s, h)\ [[\![S]\!]\mu]\ (s', h') \text{ and } (s_1, h_1)\ [[\![S]\!]\mu]\ (s_1', h_1') \text{ and}$$
$$(s, h)\ \&\ (s_1, h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_0.$$

Then there exists η' *over* h' *extending* η, *there exists* η_1' *over* h_1' *extending* η_1, *and there exists* β' *over* $h'\&h_1'$ *extending* β *over* $h\&h_1$, *such that*

(1a) $(s, h) \rightarrow (s', h')\ \models_{\eta'}\ X$

(1b) $(s_1, h_1) \rightarrow (s_1', h_1')\ \models_{\eta_1'}\ X$

(2) $(s', h')\ \&\ (s_1', h_1')\ \models_{\beta',\eta',\eta_1'}\ \phi$

Conditions **(1a)** and **(1b)** say that X is a sound overapproximation of the abstract entities modified/created in S when its execution changes the state from (s, h) to (s', h'), or from (s_1, h_1) to (s_1', h_1'). Condition **(2)** says, under the assumption that precondition ϕ_0 holds for the initial pair of states (s, h) and (s_1, h_1), that the postcondition ϕ holds for the modified states (s', h') and (s_1', h_1'). Note that these conditions hold vacuously in case of non-termination, or run-time error (since then, states (s', h') and (s_1', h_1') would not exist).

Conjunction Rule not Sound in Semantic Model. It may be the case that $\mu \models \{\phi_0\}\ S\ \{\phi_1\}\ [X]$ and $\mu \models \{\phi_0\}\ S\ \{\phi_2\}\ [X]$ hold separately, but $\mu \models \{\phi_0\}\ S\ \{\phi_1 \wedge \phi_2\}\ [X]$ does not hold. For a concrete example, consider the following program S:

$$\textbf{if } z \textbf{ then } x := \textbf{new } C;\ y := x \textbf{ else } x := \textbf{new } C;\ y := \textbf{new } C$$

Using Def. 5.1, we can semantically establish $x\ltimes$ and $y\ltimes$ separately, but *not* $x\ltimes \wedge y\ltimes$. To see this, consider the initial states (s, h) and (s_1, h_1), evolving into states (s', h') and (s_1', h_1'). Our goal is to find β' extending β such that $(s'\ x)\beta(s_1'\ x)$ and $(s'\ y)\beta(s_1'\ y)$; this is trivial if $s(z)$ and $s_1(z)$ assume the same truth value, so assume that $s(z) \in \mathsf{True}$ but $s_1(z) \in \mathsf{False}$. Then there exists fresh location ℓ such that $s'(x) = s'(y) = \ell$, and there exists fresh locations $\ell_x \neq \ell_y$ such that $s_1'(x) = \ell_x$ and $s_1'(y) = \ell_y$. To establish $x\ltimes$, we define β' such that $\ell\,\beta'\,\ell_x$; similarly, to establish $y\ltimes$, we define β' such that $\ell\,\beta'\,\ell_y$. But to establish *both* $x\ltimes$ and $y\ltimes$, we would need $\ell\,\beta'\,\ell_x$ *and* $\ell\,\beta'\,\ell_y$, which conflicts with β' being a bijection.

5.2 Syntax-directed Rules

Table 2 gives the details of some small specifications. First note that ordinary assignment, $x := E$, is split into three cases – pure assignment, where E is an arithmetic expression; pointer assignment, where E is a variable z denoting a location; and null assignment, where E is **null**.

Next note that for a given small specification, its points-to assertions are always relevant, in that those occurring in the precondition must be established by the context, whereas the agreement assertions may or may not be relevant, depending on whether those occurring in the precondition are established by the context. Therefore certain specifications should be read as *two* specifications (for space reasons, we do not show both), with the "optional" agreement assertions being listed right of a semicolon. For example, [PointerAssign] should be read as the two rules: $\{z \rightsquigarrow \rho\}\ x := z\ \{x \rightsquigarrow \rho\}\ [\{x\}]$ and $\{z \rightsquigarrow \rho,\ z\ltimes\}\ x := z\ \{x \rightsquigarrow \rho,\ x\ltimes\}\ [\{x\}]$.

Due to the presence of the points-to assertion $L\ \mathit{abs}\ i$, there are also two versions of [New] and of [FieldUpd]; the latter can be written as one rule using the disjunction operator, whereas for the former we shall write both versions explicitly.

Many of the rules in Table 2 have already been motivated by means of examples in Sec. 2, so below we shall discuss only a few, and also give the rule for method calls.

The postcondition of [New] asserts that x will be at some abstract location L with $L \neq \bot$; furthermore, $x\ltimes$ always holds and x is modified and L is created. The rule mirrors the concrete semantics of **new**, where a fresh location is allocated in the heap, except that we do not require freshness of L. On the other hand, if L was not used before, i.e., the precondition contains the assertion $L\ \mathit{abs}\ 0$, we can assert that L contains a unique location afterwards, i.e., the postcondition contains the assertion $L\ \mathit{abs}\ 1$.

In the rule [FieldUpd], we are able to handle "strong update": if we can statically infer ($L\ \mathit{abs}\ 1$) that L contains only a *single* concrete heap location, then the assertions $L.f\ltimes$ and $L.f \rightsquigarrow LI$ are not needed in the precondition.

Also note that in the absence of $L\ \mathit{abs}\ 1$, the postcondition can never contain an assertion $L.f \rightsquigarrow L'$ unless the precondition contains an assertion of the form $L.f \rightsquigarrow L''$ (where $L'' \preceq L'$); this is unlike the situation for $x \rightsquigarrow L'$ which may be introduced by the logic "ex nihilo" (using, e.g., [PointerAssign]). Therefore, the precondition for the whole program may have to explicitly contain assertions like $L.f \rightsquigarrow \bot$.

Next we discuss [If], which is similar to the rule for conditionals in Hoare logic, except that in the presence of agreement assertions, some side conditions may be needed. Two cases:

(a) If ϕ_0 logically implies $x\ltimes$, then we know that in states (s, h) and (s_1, h_1), both $s(x)$ and $s_1(x)$ will have the *same* (integer) value, so the *same branch* of the conditional will be taken during evaluation. Hence, there is no indirect control flow, and thus no need for any side conditions. (In the context of security, this case amounts to the guard of the conditional being "low").

(b) Alternatively, in states (s, h) and (s_1, h_1), $s(x)$ and $s_1(x)$ may differ, causing different branches of the conditional to be taken. In this case, in order to assert $w\ltimes$ at the end of the conditional, it does not suffice to assert $w\ltimes$ at the end of each branch, since this merely says that two runs choosing the *same* branch will agree on the value of w. What we need is that:

1. w is not modified in any branch. (In the context of security, this amounts to "no write down" under a "high guard" [7]).
2. the two runs agree on the value of w *before* the conditional.

The first demand can be encoded as $\mathcal{I}(\phi) \diamond X$; the second, as $\phi_0 \blacktriangleright \mathcal{I}(\phi)$.

In addition to (1) and (2), we must demand that ϕ does not contain `ior`. For otherwise, let S be **if** h **then** S_1 **else** S_2 with $S_1 = x := y$ and $S_2 = x := z$. We can deduce (using [Conseq])

$\{y \rightsquigarrow L, z \rightsquigarrow L_1\}\ S_1\ \{x \rightsquigarrow L\ \texttt{ior}\ x \rightsquigarrow L_1\}\ [\{x\}]$ and
$\{y \rightsquigarrow L, z \rightsquigarrow L_1\}\ S_2\ \{x \rightsquigarrow L\ \texttt{ior}\ x \rightsquigarrow L_1\}\ [\{x\}]$.

From this we could deduce, using an [If] without this extra side condition, that

$$\{y \rightsquigarrow L, z \rightsquigarrow L_1\}\ S\ \{x \rightsquigarrow L\ \texttt{ior}\ x \rightsquigarrow L_1\}\ [\{x\}]$$

which if $L \diamond L_1$ does not hold semantically, as can be seen by considering two initial states that select different branches.

$$[\mathsf{Assert}]\quad \{\mathit{true}\}\ \mathbf{assert}\ \theta\ \{\theta\}\ [\emptyset] \qquad [\mathsf{PureAssign}]\quad \frac{\{z_1,\ldots,z_n\} = \mathrm{free}(E)}{\{\mathit{true};\ z_1\ltimes,\ldots,z_n\ltimes\}\ x := E\ \{x \rightsquigarrow \mathbf{int};\ x\ltimes\}\ [\{x\}]}$$

$$[\mathsf{NullAssign}]\quad \{\mathit{true}\}\ x := \mathbf{null}\ \{x \rightsquigarrow \bot,\ x\ltimes\}\ [\{x\}] \qquad [\mathsf{PointerAssign}]\quad \{z \rightsquigarrow \rho;\ z\ltimes\}\ x := z\ \{x \rightsquigarrow \rho;\ x\ltimes\}\ [\{x\}]$$

$$[\mathsf{FieldAcc}]\quad \begin{array}{l} \{y \rightsquigarrow \rho, \rho.f \rightsquigarrow \varrho;\ y\ltimes, \rho.f\ltimes\} \\ \qquad x := y.f \\ \{x \rightsquigarrow \varrho;\ x\ltimes\} \\ [\{x\}] \end{array} \qquad [\mathsf{FieldUpd}]\quad \begin{array}{l} \{x \rightsquigarrow \rho, y \rightsquigarrow \varrho, (\rho\ \mathit{abs}\ 1)\ \mathtt{ior}\ (\rho.f \rightsquigarrow \varrho); \\ x\ltimes, y\ltimes, (\rho\ \mathit{abs}\ 1)\ \mathtt{ior}\ (\rho.f\ltimes)\} \\ \qquad x.f := y \\ \{\rho.f \rightsquigarrow \varrho;\ \rho.f\ltimes\} \\ [\{\rho.f\}] \end{array}$$

$$\begin{array}{lllll} [\mathsf{New}] & \{\mathit{true}\} & x := \mathbf{new}\ C & \{x \rightsquigarrow \rho,\ x\ltimes\}\ [\{x,\rho\}] & \text{where } \rho \neq \bot \\ [\mathsf{New}] & \{\rho\ \mathit{abs}\ 0\} & x := \mathbf{new}\ C & \{x \rightsquigarrow \rho, \rho\ \mathit{abs}\ 1, x\ltimes\}\ [\{x,\rho\}] & \text{where } \rho \neq \bot \end{array}$$

$$[\mathsf{Seq}]\quad \frac{\{\phi_0\}\ S_1\ \{\phi_1\}\ [X_1] \qquad \{\phi_1\}\ S_2\ \{\phi\}\ [X_2]}{\{\phi_0\}\ S_1\,;S_2\ \{\phi\}\ [X_1 \cup X_2]} \qquad [\mathsf{If}]\quad \frac{\{\phi_0\}\ S_1\ \{\phi\}\ [X] \qquad \{\phi_0\}\ S_2\ \{\phi\}\ [X]}{\{\phi_0\}\ \mathbf{if}\ x\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2\ \{\phi\}\ [X]} \quad \begin{array}{ll} \text{where} & \phi_0 \blacktriangleright x\ltimes \\ \text{or} & \phi \text{ contains no } \mathtt{ior} \text{ and} \\ & \mathcal{I}(\phi) \diamond X \text{ and} \\ & \phi_0 \blacktriangleright \mathcal{I}(\phi) \end{array}$$

$$[\mathsf{While}]\quad \frac{\{\phi\}\ S\ \{\phi\}\ [X]}{\{\phi\}\ \mathbf{while}\ x\ \mathbf{do}\ S\ \{\phi\}\ [X]} \quad \begin{array}{ll} \text{where} & \phi \blacktriangleright x\ltimes \\ \text{or} & \phi \text{ contains no } \mathtt{ior} \text{ and} \\ & \mathcal{I}(\phi) \diamond X \end{array}$$

Table 2: Small specifications. A ρ (a ϱ) is a metavariable to be instantiated by an L (an LI).

Concerning the specification of a method call, $x := y.m(z)$, assume that $\mathit{type}\, y = C$ and that $\Pi(C, m)$ contains the summary $\{\psi_0\}\ \_\ \{\psi\}\ [X]$. Then, with $\phi_0 = \psi_0[y/\mathit{self}, z/\mathit{pars}(m, C)]$ and $\phi = \psi[x/\mathit{result}]$,[13] we have

$$[\mathsf{MethodCall}]\quad \Pi \vdash \{\phi_0\}\ x := y.m(z)\ \{\phi\}\ [X \cup \{x\}] \quad \begin{array}{ll} \text{where} & \phi_0 \blacktriangleright y\ltimes \\ \text{or} & \phi \text{ contains no } \mathtt{ior} \text{ and} \\ & \mathcal{I}(\phi) \diamond (X \cup \{x\}) \text{ and} \\ & \phi_0 \blacktriangleright \mathcal{I}(\phi) \end{array}$$

The side condition is similar to the one found in [If], and is needed to accommodate dynamic dispatch: if $y\ltimes$ does not hold, then the two runs in question may execute different method bodies.

5.3 Structural Rules

There are four structural[14] rules: the rule of consequence which is similar to the corresponding rule in Hoare logic, a rule for each of the disjunction operators, and then the *frame rule*.

$$[\mathsf{Conseq}]\quad \frac{\{\phi_1\}\ S\ \{\phi_2\}\ [X]}{\{\phi_1'\}\ S\ \{\phi_2'\}\ [X']} \quad \begin{array}{lll} \text{if} & \phi_1' & \blacktriangleright\ \phi_1 \\ \text{and} & \phi_2 & \blacktriangleright\ \phi_2' \\ \text{and} & X & \blacktriangleright\ X' \end{array}$$

$$[\mathsf{Ior}]\quad \frac{\{\phi_1\}\ S\ \{\phi_1'\}\ [X] \quad \{\phi_2\}\ S\ \{\phi_2'\}\ [X]}{\{\phi_1\ \mathtt{ior}\ \phi_2\}\ S\ \{\phi_1'\ \mathtt{ior}\ \phi_2'\}\ [X]}$$

[13] The notation, e.g., $\psi[x/\mathit{result}]$ denotes substitution of x for *result* in ψ.
[14] I.e., not syntax-directed.

$$[\mathsf{Uor}] \quad \frac{\{\phi_1\}\ S\ \{\phi_1'\}\ [X] \quad \{\phi_2\}\ S\ \{\phi_2'\}\ [X]}{\{\phi_1 \ \mathtt{uor}\ \phi_2\}\ S\ \{\phi_1' \ \mathtt{uor}\ \phi_2'\}\ [X]}$$

$$[\mathsf{Frame}] \quad \frac{\{\phi_1\}\ S\ \{\phi_2\}\ [X]}{\{\phi_1 \wedge \phi\}\ S\ \{\phi_2 \wedge \phi\}\ [X]} \text{ if } \phi \diamond X.$$

The frame rule is used to reason with small specifications in a larger context. For example, for a command $S_1 \,; S_2$, rule [Seq] requires the postcondition of S_1 to be the same as the precondition of S_2. As the examples in Sec. 2 depict, such a match may not always be achievable by small specifications themselves: extra assertions must be added by invoking [Frame]. This is sound provided the added assertions are disjoint from the modified abstract addresses.

As suggested by the semantic considerations in Sec. 5.1, we do *not* have a rule of conjunction like the one in Hoare logic (without heaps), i.e., we cannot derive $\{\phi_0 \wedge \phi_0'\}\ S\ \{\phi \wedge \phi'\}\ [X \cup X']$ from $\{\phi_0\}\ S\ \{\phi\}\ [X]$ and $\{\phi_0'\}\ S\ \{\phi'\}\ [X']$. To see why this would be unsound (at least in our semantic model), let S be the command $x := \textbf{new}\ C$. Then, for all L_1 and L_2, we would have $\{\mathit{true}\}\ S\ \{x \rightsquigarrow L_1\}\ [\{x\}]$ and $\{\mathit{true}\}\ S\ \{x \rightsquigarrow L_2\}\ [\{x\}]$ and by the proposed conjunction rule therefore $\{\mathit{true}\}\ S\ \{x \rightsquigarrow L_1 \wedge x \rightsquigarrow L_2\}\ [\{x\}]$. But this is clearly a semantic impossibility if $L_1 \diamond L_2$. Referring back to Def. 5.1, the issue is that with ℓ' the location created by S we can extend η into an η' with $\ell'\,\eta'\,L_1$, and also into an η' with $\ell'\,\eta'\,L_2$, but we cannot find an η' with $\ell'\,\eta'\,L_1$ *and* $\ell'\,\eta'\,L_2$.

Remarks.

- One may think that the small specifications lose information and may not be precise. For example, in [PointerAssign], why did $z \ltimes$ disappear in the postcondition? But that agreement assertion can be recovered by [Frame], since z is not modified.
- Similarly, thanks to [Conseq], the rule for field update does not lose precision. To see this, assume that $y \rightsquigarrow L_1$ and $L.f \rightsquigarrow L_2$, and that L_3 is an upper bound of L_1 and L_2 with respect to $\preceq$. Then, with $x \rightsquigarrow L$, we have, by [FieldUpd]

  $$\{x \rightsquigarrow L, L.f \rightsquigarrow L_3, y \rightsquigarrow L_3\}\ x.f := y\ \{L.f \rightsquigarrow L_3\}\ [\{L.f\}]$$

  and by [Conseq] therefore

  $$\{x \rightsquigarrow L, L.f \rightsquigarrow L_2, y \rightsquigarrow L_1\}\ x.f := y\ \{L.f \rightsquigarrow L_3\}\ [\{L.f\}]$$

  and by [Frame] therefore, since x, y are not modified, and since $\wedge$ is idempotent,

  $$\begin{array}{l}\{x \rightsquigarrow L, L.f \rightsquigarrow L_2, y \rightsquigarrow L_1\} \\ \qquad x.f := y \\ \{L.f \rightsquigarrow L_3, y \rightsquigarrow L_1, x \rightsquigarrow L\}\ [\{L.f\}]\end{array}$$

  So y is not polluted: we still have $y \rightsquigarrow L_1$ after the field update.

5.4 Soundness

Definition 5.2 (Consistent summary environment) *Say that summary environment Π is consistent wrt. class table CT if whenever $\Pi(C, m)$ contains the summary $\{\psi_0\}\ \_\ \{\psi\}\ [X]$, and S is the body of a declaration of m in C, or in any subclass of C, then $\Pi \vdash \{\psi_0\}\ S\ \{\psi\}\ [X']$ where $X = \{L.f \mid L.f \in X'\} \cup \{L \mid L \in X'\}$.*

The idea is that even if a local variable is modified by S and hence occurs in X', it should not occur in X since it is not visible outside m. On the other hand, all field updates[15] are globally visible.

[15] Since we are handling only public fields. In future work, we hope to explore issues involving information hiding through private fields.

Theorem 5.3 (Soundness) *Let Π be a summary environment consistent wrt. class table CT. For a command S, suppose $\Pi \vdash \{\phi_0\}\ S\ \{\phi\}\ [X]$. Then $[\![CT]\!] \models \{\phi_0\}\ S\ \{\phi\}\ [X]$.*

The proof will follow shortly. As a special case of the theorem, the following result shows that "the points-to analysis part is sound".

Corollary 5.4 *Let Π be a summary environment consistent wrt. class table CT. Also, let $\Pi \vdash \{\phi_0\}\ S\ \{\phi\}\ [X]$.*
Assume that with $\mu = [\![CT]\!]$, we have $(s, h)\, [[\![S]\!]\mu]\, (s', h')$ and that $(s, h) \models_\eta \phi_0$.
Then there exists η' over h' extending η over h such that $(s, h) \to (s', h') \models_{\eta'} X$ and $(s', h') \models_{\eta'} \phi$.

Proof: We know from Lemma 4.3 that with β the identity relation on $dom(h)$, we have $(s, h)\,\&\,(s, h) \models_{\beta,\eta,\eta} \phi_0$. The soundness theorem now gives us η' over h' extending η, η'_1, β' such that $(s, h) \to (s', h') \models_{\eta'} X$ and $(s', h')\,\&\,(s'_1, h'_1) \models_{\beta',\eta',\eta'_1} \phi$. The claim now follows from Lemma 4.4. ■

We now address the proof of Theorem 5.3. Due to Lemma 3.3, it is clearly sufficient to prove the following lemma:

Lemma 5.5 *Given Π is consistent wrt. CT. Assume that, with μ_n as in Definition 3.1, that*

(a) $\Pi \vdash \{\phi_0\}\ S\ \{\phi\}\ [X]$

(b) $(s, h)\, [[\![S]\!]\mu_n]\, (s', h')$ *and* $(s_1, h_1)\, [[\![S]\!]\mu_n]\, (s'_1, h'_1)$

(c) $(s, h)\,\&\,(s_1, h_1) \models_{\beta,\eta,\eta_1} \phi_0$.

Then there exists η' over h' extending η over h, η'_1 over h'_1 extending η_1 over h_1, and β' over $h'\&h'_1$ extending β over $h\&h_1$, such that

(1a) $(s, h) \to (s', h') \models_{\eta'} X$

(1b) $(s_1, h_1) \to (s'_1, h'_1) \models_{\eta'_1} X$

(2) $(s', h')\,\&\,(s'_1, h'_1) \models_{\beta',\eta',\eta'_1} \phi$

Proof: We shall prove this by an outer induction on n; for each n we shall do an inner induction on the derivation $D = \Pi \vdash \{\phi_0\}\ S\ \{\phi\}\ [X]$, and do a case analysis on the last rule applied in that derivation.

Case of Rule [Conseq].

Our assumptions are

(a) $D = \{\phi_0\}\ S\ \{\phi\}\ [X]$ because $D' = \{\phi'_0\}\ S\ \{\phi'\}\ [X']$ where $\phi_0 \blacktriangleright \phi'_0$ and $\phi' \blacktriangleright \phi$ and $X' \blacktriangleright X$.

(b) $(s, h)\, [[\![S]\!]\mu_n]\, (s', h')$ and $(s_1, h_1)\, [[\![S]\!]\mu_n]\, (s'_1, h'_1)$

(c) $(s, h)\,\&\,(s_1, h_1) \models_{\beta,\eta,\eta_1} \phi_0$.

From (c) we infer, since $\phi_0 \blacktriangleright \phi'_0$, that

$$(s, h)\,\&\,(s_1, h_1) \models_{\beta,\eta,\eta_1} \phi'_0$$

We can now apply the induction hypothesis on D', and find η' over h' extending η over h, η'_1 over h'_1 extending η_1 over h_1, and β' over $h'\&h'_1$ extending β over $h\&h_1$, such that

$$(s, h) \rightarrow (s', h') \models_{\eta'} X'$$
$$(s_1, h_1) \rightarrow (s'_1, h'_1) \models_{\eta'_1} X'$$
$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} \phi'$$

Since $\phi' \blacktriangleright \phi$, and $X' \blacktriangleright X$, we infer the desired

$$(s, h) \rightarrow (s', h') \models_{\eta'} X,\ (s_1, h_1) \rightarrow (s'_1, h'_1) \models_{\eta'_1} X,\ (s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} \phi$$

Case of Rule [Frame].

Our assumptions are:

(a) $\{\phi_0 \wedge \phi_1\}\ S\ \{\phi \wedge \phi_1\}\ [X]$ because with $\phi_1 \diamond X$ we have $D_1 = \{\phi_0\}\ S\ \{\phi\}\ [X]$

(b) $(s, h)\, [[\![S]\!]\mu_n]\, (s', h')$ and $(s_1, h_1)\, [[\![S]\!]\mu_n]\, (s'_1, h'_1)$

(c) $(s, h) \,\&\, (s_1, h_1) \models_{\beta, \eta, \eta_1} \phi_0 \wedge \phi_1$

From (c), we infer

$$(s, h) \,\&\, (s_1, h_1) \models_{\beta, \eta, \eta_1} \phi_0 \tag{1}$$

$$(s, h) \,\&\, (s_1, h_1) \models_{\beta, \eta, \eta_1} \phi_1 \tag{2}$$

Due to (1), we can apply the induction hypothesis on D_1 and find η' over h' extending η over h, η'_1 over h'_1 extending η_1 over h_1, and β' over $h' \& h'_1$ extending β over $h \& h_1$, such that

$$(s, h) \rightarrow (s', h') \models_{\eta'} X \tag{3}$$

$$(s_1, h_1) \rightarrow (s'_1, h'_1) \models_{\eta'_1} X \tag{4}$$

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} \phi$$

From (2) we now infer by Lemma 4.11

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} \phi_1$$

and thus

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} \phi \wedge \phi_1$$

which together with (3) and (4) yields the desired conclusion.

Case of Rule [Ior].

Our assumptions are

(a) $D = \{\phi_0\}\ S\ \{\phi\}\ [X]$ because with $\phi_0 = \phi'_0\ \texttt{ior}\ \phi''_0$ and $\phi = \phi'\ \texttt{ior}\ \phi''$ we have $D_1 = \{\phi'_0\}\ S\ \{\phi'\}\ [X]$ and $D_2 = \{\phi''_0\}\ S\ \{\phi''\}\ [X]$

(b) $(s, h)\, [[\![S]\!]\mu_n]\, (s', h')$ and $(s_1, h_1)\, [[\![S]\!]\mu_n]\, (s'_1, h'_1)$

(c) $(s, h) \,\&\, (s_1, h_1) \models_{\beta, \eta, \eta_1} \phi_0$

Wlog., we can assume that

$$(s, h) \,\&\, (s_1, h_1) \models_{\beta, \eta, \eta_1} \phi'_0$$

By applying the induction hypothesis on D_1, we find η' over h' extending η over h, η'_1 over h'_1 extending η_1 over h_1, and β' over $h' \& h'_1$ extending β over $h \& h_1$, such that

$$(s,h) \rightarrow (s',h') \models_{\eta'} X \tag{5}$$

$$(s_1,h_1) \rightarrow (s_1',h_1') \models_{\eta_1'} X \tag{6}$$

$$(s',h') \,\&\, (s_1',h_1') \models_{\beta',\eta',\eta_1'} \phi'$$

But then also

$$(s',h') \,\&\, (s_1',h_1') \models_{\beta',\eta',\eta_1'} \phi$$

which together with (5) and (6) yields the desired conclusion.

Case of Rule [Uor].

Our assumptions are

(a) $D = \{\phi_0\}\ S\ \{\phi\}\ [X]$ because with $\phi_0 = \phi_0'\ \texttt{uor}\ \phi_0''$ and $\phi = \phi'\ \texttt{uor}\ \phi''$ we have $D_1 = \{\phi_0'\}\ S\ \{\phi'\}\ [X]$ and $D_2 = \{\phi_0''\}\ S\ \{\phi''\}\ [X]$

(b) $(s,h)\,[[\![S]\!]\mu_n]\,(s',h')$ and $(s_1,h_1)\,[[\![S]\!]\mu_n]\,(s_1',h_1')$

(c) $(s,h) \,\&\, (s_1,h_1) \models_{\beta,\eta,\eta_1} \phi_0$

We infer, by Lemma 4.4, that $(s,h) \models_\eta \phi_0$. Wlog., we can assume that $(s,h) \models_\eta \phi_0'$. By Lemma 4.3, with id the identity on $dom(h)$, we therefore have

$$(s,h) \,\&\, (s,h) \models_{id,\eta,\eta} \phi_0'$$

We can now apply the induction hypothesis on D_1, and we find η' over h' extending η over h such that

$$(s,h) \rightarrow (s',h') \models_{\eta'} X \text{ and} \tag{7}$$
$$(s',h') \,\&\, (s',h') \models_{\_,\eta',\_} \phi'$$

and by Lemma 4.4 therefore $(s',h') \models_{\eta'} \phi'$ and thus

$$(s',h') \models_{\eta'} \phi. \tag{8}$$

Similarly, we infer that there exists η_1' over h_1' extending η_1 such that

$$(s_1,h_1) \rightarrow (s_1',h_1') \models_{\eta_1'} X \text{ and} \tag{9}$$

$$(s_1',h_1') \models_{\eta_1'} \phi. \tag{10}$$

From (8) and (10) we get

$$(s',h') \,\&\, (s_1',h_1') \models_{\beta,\eta',\eta_1'} \phi$$

which together with (7) and (9) yields the desired conclusion.

Case of Rule [Seq].

Our assumptions are

(a) $\{\phi_0\}\ S_1\,;S_2\ \{\phi\}\ [X]$ because with $X = X_1 \cup X_2$, $D_1 = \{\phi_0\}\ S_1\ \{\phi_1\}\ [X_1]$ and $D_2 = \{\phi_1\}\ S_2\ \{\phi\}\ [X_2]$

(b1) $(s,h)\,[[\![S_1\,;S_2]\!]\mu_n]\,(s',h')$ because $(s,h)\,[[\![S_1]\!]\mu_n]\,(s'',h'')$ and $(s'',h'')\,[[\![S_2]\!]\mu_n]\,(s',h')$

(b2) $(s_1,h_1)\,[[\![S_1\,;S_2]\!]\mu_n]\,(s_1',h_1')$ because $(s_1,h_1)\,[[\![S_1]\!]\mu_n]\,(s_1'',h_1'')$ and $(s_1'',h_1'')\,[[\![S_2]\!]\mu_n]\,(s_1',h_1')$

(c) $(s,h) \,\&\, (s_1,h_1) \models_{\beta,\eta,\eta_1} \phi_0$

We can apply the induction hypothesis on D_1, and find η'' over h'' extending η over h, η_1'' over h_1'' extending η_1 over h_1, and β'' over $h''\&h_1''$ extending β over $h\&h_1$, such that

$$(s,h) \to (s'',h'') \models_{\eta''} X_1 \text{ and } (s_1,h_1) \to (s_1'',h_1'') \models_{\eta_1''} X_1 \tag{11}$$

$$(s'',h'') \,\&\, (s_1'',h_1'') \models_{\beta'',\eta'',\eta_1''} \phi_1 \tag{12}$$

Using (12), we can now apply the induction hypothesis on D_2, and find η' over h' extending η'' over h'', η_1' over h_1' extending η_1'' over h_1'', and β' over $h'\&h_1'$ extending β'' over $h''\&h_1''$, such that

$$\begin{aligned}&(s'',h'') \to (s',h') \models_{\eta'} X_2 \text{ and}\\ &(s_1'',h_1'') \to (s_1',h_1') \models_{\eta_1'} X_2\end{aligned} \tag{13}$$

$$(s',h') \,\&\, (s_1',h_1') \models_{\beta',\eta',\eta_1'} \phi \tag{14}$$

Lemmas 4.8 and 4.9 show that η' over h' extends η over h, that η_1' over h_1' extends η_1 over h_1, and that β' over $h'\&h_1'$ extends β over $h\&h_1$. And Lemma 4.10 shows (using (11) and (13)) that we do indeed have

$$(s,h) \to (s',h') \models_{\eta'} X \text{ and } (s_1,h_1) \to (s_1',h_1') \models_{\eta_1'} X$$

which together with (14) is what we want.

Case of Rule [If].

Our assumptions are

(a) $\{\phi_0\}$ **if** x **then** S_1 **else** S_2 $\{\phi\}$ $[X]$ because $D_1 = \{\phi_0\}\ S_1\ \{\phi\}\ [X]$ and $D_2 = \{\phi_0\}\ S_2\ \{\phi\}\ [X]$ and where either $\phi_0 \blacktriangleright x\ltimes$ or

ϕ does not contain `ior`
and $\mathcal{I}(\phi) \diamond X$
and $\phi_0 \blacktriangleright \mathcal{I}(\phi)$

(b1) $(s,h)\,[[\![\textbf{if } x \textbf{ then } S_1 \textbf{ else } S_2]\!]\mu_n]\,(s',h')$

(b2) $(s_1,h_1)\,[[\![\textbf{if } x \textbf{ then } S_1 \textbf{ else } S_2]\!]\mu_n]\,(s_1',h_1')$

(c) $(s,h) \,\&\, (s_1,h_1) \models_{\beta,\eta,\eta_1} \phi_0$

Concerning (b1), wlog. we may assume that $s(x) \in \mathsf{True}$. Thus $(s,h)\,[[\![S_1]\!]\mu_n]\,(s',h')$.

There are now two subcases:

Subcase $s_1(x) \in \mathsf{True}$. Thus $(s_1,h_1)\,[[\![S_1]\!]\mu_n]\,(s_1',h_1')$. Then we can apply the induction hypothesis on D_1, and find η' over h' extending η over h, η_1' over h_1' extending η_1 over h_1, and β' over $h'\&h_1'$ extending β over $h\&h_1$, such that

$$\begin{aligned}&(s,h) \to (s',h') \models_{\eta'} X\\ &(s_1,h_1) \to (s_1',h_1') \models_{\eta_1'} X\\ &(s',h') \,\&\, (s_1',h_1') \models_{\beta',\eta',\eta_1'} \phi\end{aligned}$$

which amounts to the desired result.

Subcase $s_1(x) \in \mathsf{False}$. Thus $(s_1,h_1)\,[[\![S_2]\!]\mu_n]\,(s_1',h_1')$. By Lemma 4.4, our assumption (c) entails

$$(s,h) \models_{\eta} \phi_0$$

so by Lemma 4.3, with id the identity on $dom(h)$, we infer that

$$(s,h) \,\&\, (s,h) \models_{id,\eta,\eta} \phi_0.$$

Applying the induction hypothesis on D_1, we therefore get η' over h' extending η over h such that

$$(s,h) \to (s',h') \models_{\eta'} X \tag{15}$$

$$(s',h') \,\&\, (s',h') \models_{\_,\eta',\_} \phi$$

where the latter by Lemma 4.4 implies

$$(s', h') \models_{\eta'} \phi. \tag{16}$$

Similarly, by applying the induction hypothesis on D_2, we get η'_1 over h'_1 extending η_1 over h_1 such that

$$(s_1, h_1) \to (s'_1, h'_1) \models_{\eta'_1} X \tag{17}$$

$$(s'_1, h'_1) \models_{\eta'_1} \phi \tag{18}$$

Finally, define $\beta' = \beta$.

Observe that it cannot be the case that $\phi_0 \blacktriangleright x\ltimes$, for then by (c) we would have $(s, h) \,\&\, (s_1, h_1) \models_{\beta,\eta,\eta_1} x\ltimes$ and hence $(s\ x)\,\beta\,(s_1\ x)$ which contradicts $s(x) \in \mathsf{True}$ and $s_1(x) \in \mathsf{False}$. By (a), we infer that

$$\phi \text{ does not contain } \texttt{ior} \tag{19}$$

$$\mathcal{I}(\phi) \diamond X \tag{20}$$

$$\phi_0 \blacktriangleright \mathcal{I}(\phi) \tag{21}$$

From our assumption (c) we infer, using (21), that

$$(s, h) \,\&\, (s_1, h_1) \models_{\beta,\eta,\eta_1} \mathcal{I}(\phi).$$

Then by Lemma 4.11, using (15) and (17) and (20), we get

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta,\eta',\eta'_1} \mathcal{I}(\phi)$$

which by Lemma 4.5 (and Fact 4.1), using (16) and (18) and (19), gives

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta,\eta',\eta'_1} \phi$$

which together with (15) and (17) yields the desired conclusion.

Case of Rule [While].

Our assumptions are

(a) $\{\phi\}$ **while** x **do** S $\{\phi\}$ $[X]$ because $D_1 = \{\phi\}\ S\ \{\phi\}\ [X]$ where either $\phi \blacktriangleright x\ltimes$ or

ϕ does not contain $\texttt{ior}$ and
$\mathcal{I}(\phi) \diamond X$

(b) $(s, h)\,[[\![\textbf{while } x \textbf{ do } S]\!]\mu_n]\,(s', h')$ and
$(s_1, h_1)\,[[\![\textbf{while } x \textbf{ do } S]\!]\mu_n]\,(s'_1, h'_1)$

(c) $(s, h) \,\&\, (s_1, h_1) \models_{\beta,\eta,\eta_1} \phi$

By the semantics of **while**, using that clearly $R_i \subseteq R_{i+1}$ holds for all i, there exists n such that

$(s, h)\, R_n\, (s', h')$ and
$(s_1, h_1)\, R_n\, (s'_1, h'_1)$

It is thus sufficient to prove the following claim for all s, h, s_1, h_1, β, η, η_1: if

$(s, h) \,\&\, (s_1, h_1) \models_{\beta,\eta,\eta_1} \phi$ and (22)
$(s, h)\, R_n\, (s', h')$ and
$(s_1, h_1)\, R_n\, (s'_1, h'_1)$

then there exists η' over h' extending η over h, there exists η'_1 over h'_1 extending η_1 over h_1, and there exists β' over $h'\&h'_1$ extending β over $h\&h_1$, such that

(1a) $(s, h) \to (s', h') \models_{\eta'} X$

(1b) $(s_1, h_1) \to (s_1', h_1') \models_{\eta_1'} X$

(2) $(s', h') \,\&\, (s_1', h_1') \models_{\beta', \eta', \eta_1'} \phi$

We shall now prove the claim by induction on n. For reasons of symmetry, it suffices to consider three subcases.

Subcase $s(x) \in \mathsf{False}, s_1(x) \in \mathsf{False}$. Then $s' = s, h' = h, s_1' = s_1, h_1' = h_1$. The claim now trivially follows by choosing $\eta' = \eta$, $\eta_1' = \eta_1$, and $\beta' = \beta$.

Subcase $s(x) \in \mathsf{True}, s_1(x) \in \mathsf{True}$. Then $n > 0$, and there exist s'', h'', s_1'', h_1'' such that

$$
\begin{array}{l}
(s, h) \, [[\![S]\!]\mu_n] \, (s'', h'') \\
(s'', h'') \, R_{n-1} \, (s', h') \\
(s_1, h_1) \, [[\![S]\!]\mu_n] \, (s_1'', h_1'') \\
(s_1'', h_1'') \, R_{n-1} \, (s_1', h_1')
\end{array}
$$

By applying the overall induction hypothesis on D_1, we find η'' over h'' extending η over h, η_1'' over h_1'' extending η_1 over h_1, and β'' over $h'' \& h_1''$ extending β over $h \& h_1$, such that

$$
\begin{array}{l}
(s, h) \to (s'', h'') \models_{\eta''} X \\
(s_1, h_1) \to (s_1'', h_1'') \models_{\eta_1''} X
\end{array} \tag{23}
$$

and also

$$(s'', h'') \,\&\, (s_1'', h_1'') \models_{\beta'', \eta'', \eta_1''} \phi$$

We can now apply the most recent induction hypothesis, and find η' over h' extending η'' over h'', η_1' over h_1' extending η_1'' over h_1'', and β' over $h' \& h_1'$ extending β'' over $h'' \& h_1''$, such that

$$
\begin{array}{l}
(s'', h'') \to (s', h') \models_{\eta'} X \text{ and} \\
(s_1'', h_1'') \to (s_1', h_1') \models_{\eta_1'} X
\end{array} \tag{24}
$$

and also

$$(s', h') \,\&\, (s_1', h_1') \models_{\beta', \eta', \eta_1'} \phi$$

Lemmas 4.8 and 4.9 show that η' over h' extends η over h, that η_1' over h_1' extends η_1 over h_1, and that β' over $h' \& h_1'$ extends β over $h \& h_1$. And Lemma 4.10 shows (using (23) and (24)) that we do indeed have the remaining obligations:

$$(s, h) \to (s', h') \models_{\eta'} X \text{ and } (s_1, h_1) \to (s_1', h_1') \models_{\eta_1'} X.$$

Subcase $s(x) \in \mathsf{True}, s_1(x) \in \mathsf{False}$. Then $s_1' = s_1$, $h_1' = h_1$, and there exists s'', h'' such that

$$
\begin{array}{l}
(s, h) \, [[\![S]\!]\mu_n] \, (s'', h'') \text{ and} \\
(s'', h'') \, R_{n-1} \, (s', h')
\end{array}
$$

From (22), and Lemma 4.4, we get $(s, h) \models_{\eta} \phi$ which by Lemma 4.3 implies (with id the identity on $dom(h)$)

$$(s, h) \,\&\, (s, h) \models_{id, \eta, \eta} \phi \tag{25}$$

We also get (still from Lemma 4.4)

$$(s, h) \,\&\, (s_1, h_1) \models_{\beta, \eta, \eta_1} \mathcal{I}(\phi) \tag{26}$$

$$(s_1, h_1) \models_{\eta_1} \phi \tag{27}$$

Applying the overall induction hypothesis on D_1, using (25), we therefore get η'' over h'' extending η over h such that

$(s,h) \rightarrow (s'',h'') \models_{\eta''} X$ and
$(s'',h'') \,\&\, (s'',h'') \models_{\_,\eta'',\_} \phi$

and by applying the most recent induction hypothesis on the latter judgment we get η' over h' extending η'' over h'' such that

$(s'',h'') \rightarrow (s',h') \models_{\eta'} X$ and
$(s',h') \,\&\, (s',h') \models_{\_,\eta',\_} \phi$

and thus by Lemma 4.4

$$(s',h') \models_{\eta'} \phi \tag{28}$$

Lemma 4.8 shows that η' over h' extends η over h. And Lemma 4.10 shows that

$$(s,h) \rightarrow (s',h') \models_{\eta'} X \tag{29}$$

Define $\beta' = \beta$, and $\eta'_1 = \eta_1$. Clearly,

$$(s_1,h_1) \rightarrow (s'_1,h'_1) \models_{\eta'_1} X \tag{30}$$

Observe that it cannot be the case that $\phi \blacktriangleright x\ltimes$, for then by (22) we would have $(s,h) \,\&\, (s_1,h_1) \models_{\beta,\eta,\eta_1} x\ltimes$ and hence $(s\ x)\,\beta\,(s_1\ x)$ which contradicts $s(x) \in \mathsf{True}$ and $s_1(x) \in \mathsf{False}$. Therefore, by assumption (a),

$$\phi \text{ does not contain } \texttt{ior} \tag{31}$$
$$\mathcal{I}(\phi) \diamond X \tag{32}$$

Lemma 4.11 applied to (26), (29), (30), (32) gives

$(s',h') \,\&\, (s'_1,h'_1) \models_{\beta',\eta',\eta'_1} \mathcal{I}(\phi)$

Together with (27) and (28), Lemma 4.5 (applicable due to (31)) now shows

$(s',h') \,\&\, (s'_1,h'_1) \models_{\beta',\eta',\eta'_1} \phi$

which with (29) and (30) amounts to the desired conclusion.

Case of Rule [Assert].
Our assumptions are

(a) $\{\mathit{true}\}\ \textbf{assert}\ \theta\ \{\theta\}\ [\emptyset]$

(b) $(s,h)\,[[\![\textbf{assert}\ \theta]\!]\mu_n]\,(s',h')$ and $(s_1,h_1)\,[[\![\textbf{assert}\ \theta]\!]\mu_n]\,(s'_1,h'_1)$

Here $s' = s, h' = h, s'_1 = s_1, h'_1 = h_1$. Also, we have $[\![\theta]\!](s,h)$ and $[\![\theta]\!](s_1,h_1)$.
Choose $\eta' = \eta$, $\eta'_1 = \eta_1$, and $\beta' = \beta$. Trivially,

$(s,h) \rightarrow (s',h') \models_{\eta'} [\emptyset]$ and
$(s_1,h_1) \rightarrow (s'_1,h'_1) \models_{\eta'_1} [\emptyset]$

We are left with proving

$(s',h') \,\&\, (s'_1,h'_1) \models_{\beta',\eta',\eta'_1} \theta$

that is $(s,h) \,\&\, (s_1,h_1) \models_{\beta,\eta,\eta_1} \theta$, which amounts to our assumptions $[\![\theta]\!](s,h)$ and $[\![\theta]\!](s_1,h_1)$.

Case of Rule [$\mathsf{PureAssign}$].
We are given the derivation

$\{\mathit{true};\ z_1\ltimes,\ldots,z_n\ltimes\}\ x := E\ \{x \rightsquigarrow \textbf{int};\ x\ltimes\}\ [\{x\}]$
where $z_1,\ldots,z_n$ are the free variables in E

Also, we assume

$$(s, h)\,[\llbracket x := E\rrbracket\mu_n]\,(s', h') \text{ and}$$
$$(s_1, h_1)\,[\llbracket x := E\rrbracket\mu_n]\,(s'_1, h'_1)$$

where $h' = h$ and $h'_1 = h_1$. And, there exists c such that $\llbracket E\rrbracket s = c$ and $s' = [s \mid x \mapsto c]$; and, there exists c_1 such that $\llbracket E\rrbracket s_1 = c_1$ and $s'_1 = [s_1 \mid x \mapsto c_1]$.

Choose $\eta' = \eta$, $\eta'_1 = \eta_1$, and $\beta' = \beta$. Trivially,

$$(s, h) \to (s', h') \models_{\eta'} [\{x\}] \text{ and}$$
$$(s_1, h_1) \to (s'_1, h'_1) \models_{\eta'_1} [\{x\}] \text{ and}$$
$$(s', h')\,\&\,(s'_1, h'_1) \models_{\beta',\eta',\eta'_1} x \rightsquigarrow \mathbf{int}$$

In the case where there are no agreement assertions, we are done. Otherwise, our assumptions entail that for all z in free(E) we have

$$(s, h)\,\&\,(s_1, h_1) \models_{\beta,\eta,\eta_1} z\ltimes$$

and we must prove

$$(s', h')\,\&\,(s'_1, h'_1) \models_{\beta',\eta',\eta'_1} x\ltimes$$

But for all z in free(E), we have $(s\ z)\,\beta\,(s_1\ z)$ and thus (since z has integer type) $s(z) = s_1(z)$. Therefore, since $\llbracket E\rrbracket$ is deterministic, $c = c_1$. So $s'(x) = s'_1(x)$, and thus $(s'x)\,\beta\,(s'_1x)$ as desired.

Case of Rule [PointerAssign].

We are given the derivation

$$\{z \rightsquigarrow L;\ z\ltimes\}\ x := z\ \{x \rightsquigarrow L;\ x\ltimes\}\ [\{x\}]$$

Also, we assume

$$(s, h)\,[\llbracket x := z\rrbracket\mu_n]\,(s', h') \text{ and}$$
$$(s_1, h_1)\,[\llbracket x := z\rrbracket\mu_n]\,(s'_1, h'_1)$$

where $s' = [s \mid x \mapsto s(z)]$ and $s'_1 = [s_1 \mid x \mapsto s_1(z)]$, and $h' = h$ and $h'_1 = h_1$.

Our assumption entails

$$(s, h) \models_{\eta} z \rightsquigarrow L$$

from which we infer $(sz)\,\eta\,L$. Similarly, we have $(s_1z)\,\eta_1\,L$.

Choose $\eta' = \eta$, $\eta'_1 = \eta_1$, and $\beta' = \beta$. Obviously,

$$(s, h) \to (s', h') \models_{\eta'} \{x\} \text{ and}$$
$$(s_1, h_1) \to (s'_1, h'_1) \models_{\eta'_1} \{x\}.$$

To prove

$$(s', h')\,\&\,(s'_1, h'_1) \models_{\beta',\eta',\eta'_1} x \rightsquigarrow L$$

for reasons of symmetry it is sufficient to show

$$(s', h') \models_{\eta'} x \rightsquigarrow L$$

This follows since $s'(x) = s(z)$ and $(sz)\,\eta\,L$.

In the case where there are no agreement assertions, we are done. Otherwise, our assumptions entail

$$(s, h)\,\&\,(s_1, h_1) \models_{\beta,\eta,\eta'} z\ltimes$$

implying $(sz)\,\beta\,(s_1z)$, and we must prove

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} x \ltimes$$

But this follows since $s'(x) = s(z)$ and $s_1(z) = s'_1(x)$.

Case of Rule [NullAssign]. We are given the derivation

$$\{true\}\ x := \mathbf{null}\ \{x \rightsquigarrow \bot;\ x \ltimes\}\ [\{x\}]$$

Also, we assume

$$(s, h)\,[[\![x := \mathbf{null}]\!]\mu_n]\,(s', h') \text{ and}$$
$$(s_1, h_1)\,[[\![x := \mathbf{null}]\!]\mu_n]\,(s'_1, h'_1)$$

where $s' = [s \mid x \mapsto nil]$ and $s'_1 = [s_1 \mid x \mapsto nil]$, and where $h' = h$ and $h'_1 = h_1$.

Choose $\eta' = \eta$, $\eta'_1 = \eta_1$, and $\beta' = \beta$. Obviously,

$$(s, h) \to (s', h') \models_{\eta'} \{x\} \text{ and}$$
$$(s_1, h_1) \to (s'_1, h'_1) \models_{\eta'_1} \{x\}$$

To prove

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} x \rightsquigarrow \bot$$

for reasons of symmetry it is sufficient to show

$$(s', h') \models_{\eta'} x \rightsquigarrow \bot$$

This follows since $s'(x) = nil$ and $nil\,\eta\,\bot$.

To prove

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} x \ltimes$$

we must show that $(s'x)\,\beta\,(s'_1 x)$. But this follows since $nil\,\beta\,nil$.

Case of Rule [FieldAcc]. We are given the derivation

$$\begin{array}{l} \{z \rightsquigarrow L, L.f \rightsquigarrow LI;\ z \ltimes, L.f \ltimes\} \\ \qquad x := z.f \\ \{x \rightsquigarrow LI;\ x \ltimes\} \\ [\{x\}] \end{array}$$

Also, we assume

$$(s, h)\,[[\![x := z.f]\!]\mu_n]\,(s', h') \text{ and}$$
$$(s_1, h_1)\,[[\![x := z.f]\!]\mu_n]\,(s'_1, h'_1)$$

where $h' = h, h'_1 = h_1$, and there exists $\ell \in Loc$ such that

$$s(z) = \ell \text{ and}$$
$$s' = [s \mid x \mapsto h\ell f]$$

and there exists $\ell_1 \in Loc$ such that

$$s_1(z) = \ell_1 \text{ and}$$
$$s'_1 = [s_1 \mid x \mapsto h_1 \ell_1 f]$$

Our assumption entails

$$(s, h) \models_{\eta} z \rightsquigarrow L \text{ and}$$
$$(s, h) \models_{\eta} L.f \rightsquigarrow LI$$

from which we infer $\ell\,\eta\,L$ and subsequently $(h\,\ell f)\,\eta\,LI$. Similarly, we have $\ell_1\,\eta_1\,L$ and $(h_1\ell_1 f)\,\eta_1\,LI$.

Choose $\eta' = \eta$, $\eta'_1 = \eta_1$, and $\beta' = \beta$. Trivially,

$$(s, h) \to (s', h') \models_{\eta'} \{x\} \text{ and}$$
$$(s_1, h_1) \to (s'_1, h'_1) \models_{\eta'_1} \{x\}.$$

To prove

$$(s', h')\,\&\,(s'_1, h'_1) \models_{\beta',\eta',\eta'_1} x \rightsquigarrow LI$$

for reasons of symmetry it is sufficient to show

$$(s', h') \models_{\eta'} x \rightsquigarrow LI$$

This follows since $s'(x) = h\,\ell f$ and $(h\,\ell f)\,\eta\,LI$.

In the case where there are no agreement assertions, we are done. Otherwise, our assumptions entail

$$(s, h)\,\&\,(s_1, h_1) \models_{\beta,\eta,\eta'} z\ltimes \tag{33}$$
$$(s, h)\,\&\,(s_1, h_1) \models_{\beta,\eta,\eta'} L.f\ltimes \tag{34}$$

and we must prove

$$(s', h')\,\&\,(s'_1, h'_1) \models_{\beta',\eta',\eta'_1} x\ltimes$$

From (33) we get $(sz)\,\beta\,(s_1 z)$, that is $\ell\,\beta\,\ell_1$. Since $\ell\,\eta\,L$ and $\ell_1\,\eta_1\,L$, from (34) we infer that

$$(h\ell f)\,\beta\,(h_1\ell_1 f)$$

which amounts to the desired $(s'x)\,\beta\,(s'_1 x)$.

Case of Rule [New].

We are given the derivation

$$\{true\}\ x := \textbf{new}\ C\ \{x \rightsquigarrow L_0,\ x\ltimes\}\ [\{x, L_0\}]$$

with $L_0 \neq \bot$. Also, we assume

$$(s, h)\,[[\![x := \textbf{new}\ C]\!]\mu_n]\,(s', h') \text{ and}$$
$$(s_1, h_1)\,[[\![x := \textbf{new}\ C]\!]\mu_n]\,(s'_1, h'_1)$$

That is, there exists ℓ' with $type\,\ell' = C$ such that

$$s' = [s \mid x \mapsto \ell'] \text{ and}$$
$$h' = [h \mid \ell' \mapsto defaults]$$

and there exists ℓ'_1 with $type\,\ell'_1 = C$ such that

$$s'_1 = [s_1 \mid x \mapsto \ell'_1] \text{ and}$$
$$h'_1 = [h_1 \mid \ell'_1 \mapsto defaults]$$

where $defaults$ maps f to $default(type\,f)$. Also, ℓ' does not occur in s or in h, and ℓ'_1 does not occur in s_1 or in h_1.

Choose $\eta' = \eta \cup \{(\ell', L) \mid L_0 \preceq L\}$, and $\eta'_1 = \eta_1 \cup \{(\ell'_1, L) \mid L_0 \preceq L\}$. To see that η' is a valid extraction function, we shall rule out three scenarios:

- There exists ℓ, and $L \preceq L_1$, such that $\ell\,\eta'\,L$ but not $\ell\,\eta'\,L_1$. Since η is a valid extraction function, we infer that $\ell = \ell'$, in which case we from $\ell\,\eta'\,L$ deduce $L_0 \preceq L$. But then also $L_0 \preceq L_1$, showing that $\ell\,\eta'\,L_1$, contradicting our assumption.

- With $L_1 \diamond L_2$, there exists ℓ such that $\ell\,\eta'\,L_1$ and $\ell\,\eta'\,L_2$. Since η is a valid extraction function, we infer that $\ell = \ell'$, so $L_0 \preceq L_1$ and $L_0 \preceq L_2$ from which we infer (by our assumptions about abstract locations) first $L_0 \diamond L_2$ and next $L_0 \diamond L_0$, yielding a contradiction since $L_0 \neq \bot$.

- There exists ℓ with $\ell\,\eta'\,\bot$. Since η is a valid extraction function, we infer that $\ell = \ell'$. But then $L_0 \preceq \bot$, which contradicts $L_0 \neq \bot$.

Similarly, we can show that η'_1 is a valid extraction function. To see that η' extends η, note that for $\ell \in dom(h)$ we have $\ell' \neq \ell$ and therefore $\ell\,\eta\,L$ iff $\ell\,\eta'\,L$. Similarly, we infer that η'_1 extends η_1.

Also, we define $\beta' = \beta \cup \{(\ell', \ell'_1)\}$. Note that, since $\ell' \notin dom(h)$ and $\ell'_1 \notin dom(h_1)$, it holds that

$$\beta = \{(\ell, \ell_1) \in \beta' \mid \ell \in dom(h) \text{ or } \ell_1 \in dom(h_1)\}$$

and hence β' extends β. Also, if $\ell\,\beta'\,\ell_1$ then for all L we have $\ell\,\eta'\,L$ iff $\ell_1\,\eta'_1\,L$.

Due to our definition (Def. 3.5) of "modified", no $\ell.f$ is modified from h to h', and no $\ell.f$ is modified from h_1 to h'_1. Since only ℓ' is created from h to h', and only ℓ'_1 is created from h_1 to h'_1, this shows

$$(s, h) \to (s', h') \models_{\eta'} [\{x, L_0\}] \text{ and}$$
$$(s_1, h_1) \to (s'_1, h'_1) \models_{\eta'_1} [\{x, L_0\}]$$

where we have used $\ell'\,\eta'\,L_0$ and $\ell'_1\,\eta'_1\,L_0$ which also implies

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} x \rightsquigarrow L_0$$

We are left with proving

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} x \ltimes$$

But this follows since $s'(x) = \ell'$ and $\ell'\,\beta\,\ell'_1$ and $\ell'_1 = s'_1(x)$.

The case when in addition, L_0 *abs* 0 is updated to L_0 *abs* 1, is an easy modification of the above: we know that

$$(s, h) \,\&\, (s_1, h_1) \models_{\beta, \eta, \eta_1} L_0 \; abs \; 0$$

from which we infer that $\ell\,\eta\,L_0$ holds for no $\ell \in dom(h)$, and that $\ell_1\,\eta_1\,L_0$ holds for no $\ell_1 \in dom(h_1)$. Therefore, $\ell\,\eta'\,L_0$ implies that $\ell = \ell'$; similarly, $\ell_1\,\eta'_1\,L_0$ implies that $\ell_1 = \ell'_1$ (with ℓ' and ℓ'_1 as defined in the main case). We conclude the desired extra postcondition:

$$(s', h') \,\&\, (s'_1, h'_1) \models_{\beta', \eta', \eta'_1} L_0 \; abs \; 1$$

<u>Case of Rule [FieldUpd].</u>

We are given the derivation

$$\begin{array}{l} \{x \rightsquigarrow L, z \rightsquigarrow LI, L \; abs \; 1 \; \texttt{ior} \; L.f \rightsquigarrow LI; \\ \quad x \ltimes, z \ltimes, L \; abs \; 1 \; \texttt{ior} \; L.f \ltimes\} \\ \qquad x.f := z \\ \{L.f \rightsquigarrow LI; \; L.f \ltimes\} \\ [\{L.f\}] \end{array}$$

Also, we assume

$$(s, h) \, [\![\![x.f := z]\!]\!\mu_n] \, (s', h') \text{ and}$$
$$(s_1, h_1) \, [\![\![x.f := z]\!]\!\mu_n] \, (s'_1, h'_1)$$

Let $\ell' = s(x)$, and let $\ell'_1 = s_1(x)$. Thus, $s = s'$, $dom(h) = dom(h')$, $s_1 = s'_1$, $dom(h_1) = dom(h'_1)$, and

$$h' = [h \mid \ell'.f \mapsto s(z)]$$
$$h'_1 = [h_1 \mid \ell'_1.f \mapsto s_1(z)]$$

Our assumption entails

$$(s,h) \models_\eta x \rightsquigarrow L \tag{35}$$

$$(s,h) \models_\eta z \rightsquigarrow LI \tag{36}$$

$$(s,h) \models_\eta L.f \rightsquigarrow LI \text{ or } (s,h) \models_\eta L\ abs\ 1 \tag{37}$$

From (35) we infer $\ell' \,\eta\, L$; similarly we infer $\ell'_1 \,\eta_1\, L$. Choose $\eta' = \eta$, $\eta'_1 = \eta_1$, and $\beta' = \beta$. It is easy to see that

$$(s,h) \rightarrow (s',h') \models_{\eta'} \{L.f\} \text{ and}$$
$$(s_1,h_1) \rightarrow (s'_1,h'_1) \models_{\eta'_1} \{L.f\}.$$

To prove

$$(s',h') \,\&\, (s'_1,h'_1) \models_{\beta',\eta',\eta'_1} L.f \rightsquigarrow LI$$

for reasons of symmetry it is sufficient to show

$$(s',h') \models_{\eta'} L.f \rightsquigarrow LI$$

So consider $\ell \in dom(h')$ with $\ell \,\eta\, L$; we must show $(h'\ell f) \,\eta\, LI$. We have two subcases:

Subcase $\ell = \ell'$. Then $h'\ell f = s(z)$, and the claim follows from (36).

Subcase $\ell \neq \ell'$. Then $h'\ell f = h\ell f$. Observe that $(s,h) \models_\eta L\ abs\ 1$ does not hold, so from (37) we infer that

$$(s,h) \models_\eta L.f \rightsquigarrow LI$$

from which the claim follows.

In the case where there are no agreement assertions, we are done. Otherwise, our assumptions entail

$$(s,h) \,\&\, (s_1,h_1) \models_{\beta,\eta,\eta_1} x \ltimes \tag{38}$$

$$(s,h) \,\&\, (s_1,h_1) \models_{\beta,\eta,\eta_1} z \ltimes \tag{39}$$

$$(s,h) \,\&\, (s_1,h_1) \models_{\beta,\eta,\eta_1} L\ abs\ 1 \texttt{ ior } L.f \ltimes \tag{40}$$

and we must prove

$$(s',h') \,\&\, (s'_1,h'_1) \models_{\beta',\eta',\eta'_1} L.f \ltimes$$

So consider $\ell \in dom(h')$ and $\ell_1 \in dom(h'_1)$ with $\ell \,\beta\, \ell_1$ and $\ell \,\eta\, L$ (thus also $\ell_1 \,\eta_1\, L$). We must show that $(h'\ell f) \,\beta\, (h'_1\ell_1 f)$. From (38) we infer that $(sx) \,\beta\, (s_1 x)$, that is, $\ell' \,\beta\, \ell'_1$. We have two subcases:

Subcase $\ell = \ell'$. Since β is a bijection, $\ell_1 = \ell'_1$; the claim is thus that $(sz) \,\beta\, (s_1 z)$, but this follows from (39).

Subcase $\ell \neq \ell'$. Since β is a bijection, $\ell_1 \neq \ell'_1$; the claim is thus that $(h\ell f) \,\beta\, (h_1\ell_1 f)$. This follows from (40), using that $L\ abs\ 1$ cannot hold (since $\ell \,\eta\, L$ and $\ell' \,\eta\, L$).

Case of Rule [MethodCall].

We are given the derivation

$$\Pi \vdash \{\phi_0\}\ x := y.m(w)\ \{\phi\}\ [X \cup \{x\}]$$

because with $C_0 = type\, y$ it holds that $\Pi(C_0, m)$ contains the summary

$$\{\psi_0\}\ \_\ \{\psi\}\ [X]$$

Here $\phi_0 = \psi_0[y/\mathit{self}, w/z]$ and $\phi = \psi[x/\mathit{result}]$, where $z = \mathit{pars}(m, C_0)$.

We also assume that

$$(s, h)\,[\llbracket x := y.m(w)\rrbracket\mu_n]\,(s', h') \text{ and}$$
$$(s_1, h_1)\,[\llbracket x := y.m(w)\rrbracket\mu_n]\,(s_1', h_1')$$

Let $s(y) = \ell$ with $\mathit{type}\,\ell = C$, and let $s_1(y) = \ell_1$ with $\mathit{type}\,\ell_1 = C_1$. Since we assume our program is well-typed, we know that C and C_1 are subclasses of C_0. From the semantics of method calls, there exists v, v_1 such that

$$s' = [s \mid x \mapsto v] \text{ and } s_1' = [s_1 \mid x \mapsto v_1]$$

and with

$$s'' = [z \mapsto s(w), \mathit{self} \mapsto \ell] \text{ and } s_1'' = [z \mapsto s_1(w), \mathit{self} \mapsto \ell_1]$$

we have

$$(s'', h)\,\mu_n(C, m)\,(v, h') \tag{41}$$

$$(s_1'', h_1)\,\mu_n(C_1, m)\,(v_1, h_1') \tag{42}$$

With S the body of the declaration of m in C, and with S_1 the body of the declaration of m in C_1, we deduce from Π being consistent that there exists X', X_1' with

$$X = \{L.f \mid L.f \in X'\} \cup \{L \mid L \in X'\}$$
$$X = \{L.f \mid L.f \in X_1'\} \cup \{L \mid L \in X_1'\}$$

such that

$$\Pi \vdash \{\psi_0\}\ S\ \{\psi\}\ [X'] \tag{43}$$

$$\Pi \vdash \{\psi_0\}\ S_1\ \{\psi\}\ [X_1']$$

From (41) and (42), and the definition of μ_n, we infer that there exists s''' with $v = s'''(\mathit{result})$ and s_1''' with $v_1 = s_1'''(\mathit{result})$ such that

$$(s'', h)\,[\llbracket S\rrbracket\mu_{n-1}]\,(s''', h') \tag{44}$$

$$(s_1'', h_1)\,[\llbracket S_1\rrbracket\mu_{n-1}]\,(s_1''', h_1') \tag{45}$$

Finally, we assume that

$$(s, h)\,\&\,(s_1, h_1)\ \models_{\beta,\eta,\eta_1}\ \phi_0 \tag{46}$$

from which we infer by Lemma 4.2 that

$$(s'', h)\,\&\,(s_1'', h_1)\ \models_{\beta,\eta,\eta_1}\ \psi_0 \tag{47}$$

There are now two subcases.

The case where $\phi_0 \blacktriangleright y\ltimes$. Then we infer from (46) that $(s\,y)\,\beta\,(s_1\,y)$ and hence $C = C_1$, implying $S = S_1$. Therefore we can apply the outermost induction hypothesis on (43), (44), (45), (47), giving us η' over h' extending η over h, η_1' over h_1' extending η_1 over h_1, and β' over $h'\&h_1'$ extending β over $h\&h_1$, such that

$$\begin{array}{l}(s'', h) \to (s''', h')\ \models_{\eta'}\ X' \\ (s_1'', h_1) \to (s_1''', h_1')\ \models_{\eta_1'}\ X' \\ (s''', h')\,\&\,(s_1''', h_1')\ \models_{\beta',\eta',\eta_1'}\ \psi\end{array} \tag{48}$$

Since $s'''(\mathit{result}) = s'(x)$, and $s_1'''(\mathit{result}) = s_1'(x)$, Lemma 4.2 tells us that

$$(s', h')\,\&\,(s'_1, h'_1)\ \models_{\beta',\eta',\eta'_1}\ \phi$$

which is part of the desired conclusion. We are left with showing that

$$(s, h) \to (s', h')\ \models_{\eta'}\ X \cup \{x\}$$

(and its symmetric counterpart). So first consider y modified from s to s'; we infer $y = x$ so $y \in X \cup \{x\}$. Next consider $\ell.f$ modified from h to h' (the case where ℓ is created from h to h' is similar); from (48) we infer that there exists L with $\ell\,\eta'\,L$ such that $L.f \in X'$. But then also $L.f \in X \cup \{x\}$, as desired.

The case where $\phi_0 \blacktriangleright y\ltimes$ does not hold. Then the side conditions of [MethodCall] tell us that

$$\phi \text{ contains no } \texttt{ior} \tag{49}$$

$$\mathcal{I}(\phi) \diamond (X \cup \{x\}) \tag{50}$$

$$\phi_0 \blacktriangleright \mathcal{I}(\phi) \tag{51}$$

From (47) we infer, using Lemmas 4.4 and 4.3, that with id the identity on $dom(h)$ it holds that

$$(s'', h)\,\&\,(s'', h)\ \models_{id,\eta,\eta}\ \psi_0.$$

Applying the outermost induction hypothesis on (44) and (43), we therefore get η' over h' extending η over h such that

$$(s'', h) \to (s''', h')\ \models_{\eta'}\ X' \tag{52}$$

$$(s''', h')\,\&\,(s''', h')\ \models_{\_,\eta',\_}\ \psi$$

where the latter by Lemma 4.2 implies

$$(s', h')\,\&\,(s', h')\ \models_{\_,\eta',\_}\ \phi$$

and by Lemma 4.4 further

$$(s', h')\ \models_{\eta'}\ \phi. \tag{53}$$

Similarly, we get η'_1 over h'_1 extending η_1 over h_1 such that

$$(s''_1, h_1) \to (s'''_1, h'_1)\ \models_{\eta'_1}\ X'_1$$

$$(s'_1, h'_1)\ \models_{\eta'_1}\ \phi. \tag{54}$$

Finally, define $\beta' = \beta$.

We shall now show

$$(s, h) \to (s', h')\ \models_{\eta'}\ X \cup \{x\} \tag{55}$$

$$(s_1, h_1) \to (s'_1, h'_1)\ \models_{\eta'_1}\ X \cup \{x\} \tag{56}$$

where for reasons of symmetry it suffices to prove (55). So first consider y modified from s to s'; we infer $y = x$ so $y \in X \cup \{x\}$. Next consider $\ell.f$ modified from h to h' (the case where ℓ is created from h to h' is similar); from (52) we infer that there exists L with $\ell\,\eta'\,L$ such that $L.f \in X'$. But then also $L.f \in X \cup \{x\}$, as desired.

From (46) and (51) we infer that

$$(s, h)\,\&\,(s_1, h_1)\ \models_{\beta,\eta,\eta_1}\ I(\phi) \tag{57}$$

Lemma 4.11 applied to (57), (55), (56), (50) now gives

$$(s', h')\,\&\,(s'_1, h'_1)\ \models_{\beta',\eta',\eta'_1}\ \mathcal{I}(\phi)$$

Together with (53) and (54), Lemma 4.5 (applicable due to (49)) now shows

$$(s', h')\,\&\,(s'_1, h'_1)\ \models_{\beta',\eta',\eta'_1}\ \phi$$

which with (55) and (56) amounts to the desired conclusion. ■

6 A Larger Example

We consider the following example due to Barnett et al. [6].

```
class C{
1.  private Hashtable ht := new Hashtable;   //cache
2.  public U m(T x){
3.     Hashtable t := self.ht;
4.     bool present := t.contains(x);
5.     if (!present){
6.        U y := costly(x);
7.        t.put(x, y); }
8.     U res := (U)t.get(x);
9.     assert (res = costly(x));
10.    result := res;  } }
```

The method m is an efficient implementation of the method *costly*, employing memoization: argument-result pairs are cached in a hash table t, with the argument as key. A call to m with some argument, x, first checks if a value exists for key x in t (lines 4, 5); if not, it is computed (line 6) and stored in t (line 7). At that point, we know that the result can be retrieved from the hash table (line 8) and returned (line 10).

We shall now argue that m is observationally pure (and hence can be used in specifications). As in Sec. 2 (for *cexp*), this involves showing (i) that *result* depends on x only; (ii) that m modifies only locations not visible to the caller.

For (i), we must show that two runs which agree on the initial value of x also agree on the final value of *result*. So let $x\ltimes$ be in the precondition, then – due to the frame rule, as x is not modified along the way – $x\ltimes$ holds after line (9), where by [Assert] we also have $res = costly(x)$. By [Conseq], this entails that $res\ltimes$ holds before line (10). By [PointerAssign], this entails that $result\ltimes$ is in the postcondition, as desired.

For (ii), assume that *Hashtable* has two fields, *key* and *val*, and that it is in abstract location L_0. The only abstract addresses modified by m are $L_0.key$ and $L_0.val$ (as well as certain local variables which are not visible to the caller, cf. Definition 5.2). The desired invisibility can then be obtained by assuming that L_0 is disjoint from all abstract locations used outside of m.

For the above to work out formally, we need method summaries such as the ones below:

$$\begin{array}{cc}
\{self \rightsquigarrow \rho_0, & \{self \rightsquigarrow \rho_0, \\
\rho_0.key \rightsquigarrow \rho_1, x \rightsquigarrow \rho_1, & \rho_0.key \rightsquigarrow \rho_1, x \rightsquigarrow \rho_1, \\
\rho_0.val \rightsquigarrow \rho_2, y \rightsquigarrow \rho_2\} & \rho_0.val \rightsquigarrow \rho_2\} \\
put & get \\
\{\} & \{result \rightsquigarrow \rho_2\} \\
[\rho_0.key, \rho_0.val] & [\emptyset]
\end{array}$$

Note that we do *not* need the summaries to contain agreement assertions. It is interesting, however, to consider how such assertions could be added in the summary for, e.g., the method *get*. Naïvely, we would expect (c.f. the method *getQ* described in Sec. 2) that if the precondition ψ_0 contains the assertions $x\ltimes$, $self\ltimes$, $\rho_0.key\ltimes$, and $\rho_0.val\ltimes$, then the postcondition ψ_1 will contain the assertion $result\ltimes$. But in general, *get* cannot be implemented so as to satisfy this summary. To see this we assume, in order to arrive at a contradiction, that S_g is the body of such an implementation. By Theorem 5.3 (and Definition 5.2), we have $\mu \models \{\psi_0\}\ S_g\ \{\psi_1\}\ \_$. Now consider two states, (s, h) and (s_1, h_1), where the key $s(x)$ (which due to $x\ltimes$ equals $s_1(x)$) is mapped by the hash table to *different* integer values. With β chosen such that β relates $s(self)$ to $s_1(self)$ but relates no other locations, we have $(s, h) \,\&\, (s_1, h_1) \models_{\beta,\eta,\eta_1} \psi_0$ (since, e.g., $\rho_0.key\ltimes$ vacuously holds). But with S_g transforming (s, h) into (s', h') and (s_1, h_1) into (s'_1, h'_1), it is *not* possible (since the integers $s'(result)$ and $s'_1(result)$ are different) to define β' such that $(s', h') \,\&\, (s'_1, h'_1) \models_{\beta',\eta',\eta'_1} result\ltimes$. This yields the desired contradiction.

To fix the above situation, we need to be more concrete about how the (hash) table is implemented. Suppose that it is a linked list, with each record containing not only a *key* and a *val* field (both integers), but also a *next* field. Then, we can implement *get* such that $result\ltimes$ is in the postcondition, provided we include $\rho_0.next\ltimes$ in the precondition ψ_0. To see this, consider as above two states, (s,h) and (s_1,h_1), with $(s,h)\,\&\,(s_1,h_1)\ \models_{\beta,\eta,\eta_1}\ \psi_0$. Since ψ_0 contains $x\ltimes$, there exists an integer k such that $s(x) = s_1(x) = k$. Wlog., we can assume that in the first state, k occurs as the third key in the list. That is, there exists locations ℓ, ℓ_1, and ℓ_2 such that $s(self) = \ell$, $h\,\ell\,next = \ell_1$, $h\,\ell_1\,next = \ell_2$, and $h\,\ell_2\,key = k$. Since ψ_0 contains $self\ltimes$, with $s_1(self) = \ell'$ we have $\ell\,\beta\,\ell'$. This entails, since ψ_0 contains $\rho_0.next\ltimes$ and we can assume $\ell\,\eta\,\rho_0$, that with $h_1\,\ell'\,next = \ell'_1$ we have $\ell_1\,\beta\,\ell'_1$; similarly we then infer that $\ell_2\,\beta\,\ell'_2$ with $h_1\,\ell'_1\,next = \ell'_2$. Since ψ_0 contains $\rho_0.key\ltimes$ and $\rho_0.val\ltimes$, we now infer that $h_1\,\ell'_2\,key = h\,\ell_2\,key = k$, and that there exists v such that $h_1\,\ell'_2\,val = h\,\ell_2\,val = v$. With (s',h') and (s'_1,h'_1) the final states, this shows the desired $s'_1(result) = v = s'(result)$.

7 Computing Postconditions

```
***
  THIS SECTION, IN PARTICULAR THE LATER PARTS, IS IN A PRETTY ROUGH STATE.

  IT WILL SOON BE SUPERSEDED BY A PAPER DEVOTED TO INFERRING ASSERTIONS
***
```

It is time to address how to decide, and implement, our logic. For that purpose, we shall along the way introduce several simplifying assumptions, some of which we state already now.

Assumption 7.1 *All disjunctions in assertions occur only within programmer assertions θ. That is, we do not use the constructs ϕ_1* `ior` *ϕ_2 and ϕ_1* `uor` *ϕ_2.*

Thus, we can view an assertion ϕ as a set (implicitly a conjunction) of primitive assertions α.

Assumption 7.2 *There are no assertions of the form L abs i.*

As a consequence of that assumption, we do not need to record which locations have been created, so we can assume that an abstract entity is just an abstract address.

Assumption 7.3 *Abstract locations form a finite complete lattice, with $\bot$ the least element and $\top$ the greatest element, where $\sqcup$ "corresponds to" set union and $\sqcap$ "corresponds to" set intersection. That is, we require that*

- *if $L = L_1 \sqcup L_2$ then $\ell\,\eta\,L$ iff $\ell\,\eta\,L_1$ or $\ell\,\eta\,L_2$*
- *if $L = L_1 \sqcap L_2$ then $\ell\,\eta\,L$ iff $\ell\,\eta\,L_1$ and $\ell\,\eta\,L_2$.*

Accordingly, we also require that if $L = L_1 \sqcup L_2$ then for all L': $L' \diamond L$ iff $L' \diamond L_1$ and $L' \diamond L_2$.

(Recall from Sec. 4 that $\bot$ approximates *nil* but no concrete heap locations; on the other hand, $\top$ approximates all concrete locations.) Thus, if $L = L_1 \sqcup L_2$, then any information about $L.f$ can be deduced from information about $L_1.f$ and $L_2.f$. This motivates the next assumption:

Assumption 7.4 *Among the abstract locations are some "irreducible" elements (we write $\mathsf{irr}(L)$ for irreducible L) such that*

- *if $L_1 \neq L_2$ are irreducible then $L_1 \diamond L_2$;*
- *for each abstract location L, there are unique irreducible elements $L_1, \ldots, L_n$ $(n \geq 0)$ such that $L = L_1 \sqcup \ldots \sqcup L_n$.*

Assumption 7.5 *There are no assertions of the form $x \rightsquigarrow$* **int** *or $L.f \rightsquigarrow$* **int***, as these are redundant (cf. footnote 4); neither are there assertions of the form $x \rightsquigarrow \top$ or $L.f \rightsquigarrow \top$.*

Normalization. It is convenient to work with assertions where all abstract locations (on the "left hand side") are irreducible and occur at most once:

Definition 7.6 *Say that ϕ is normalized iff (a) if $L.f \rightsquigarrow L' \in \phi$ then L is irreducible; (b) if $L.f \ltimes \in \phi$ then L is irreducible; (c) if $L.f \rightsquigarrow L_1 \in \phi$ and $L.f \rightsquigarrow L_2 \in \phi$ then $L_1 = L_2$; (d) if $x \rightsquigarrow L_1 \in \phi$ and $x \rightsquigarrow L_2 \in \phi$ then $L_1 = L_2$; (e) ϕ contains exactly one programmer assertion.*

It is possible to write a function *norm* that converts an assertion ϕ into a normalized assertion $\mathit{norm}(\phi)$ which is logically equivalent. We define $\mathit{norm}(\phi)$ by stipulating

$$\begin{array}{rll} x \rightsquigarrow L' \in \mathit{norm}(\phi) & \text{iff} & L' \neq \top \wedge L' = \sqcap\{L \mid x \rightsquigarrow L \in \phi\} \\ L_0.f \rightsquigarrow L' \in \mathit{norm}(\phi) & \text{iff} & L' \neq \top \wedge \mathsf{irr}(L_0) \wedge L' = \sqcap\{L \mid \exists L'_0 : (L_0 \preceq L'_0 \wedge L'_0.f \rightsquigarrow L \in \phi)\} \\ x \ltimes \in \mathit{norm}(\phi) & \text{iff} & x \ltimes \in \phi \\ L_0.f \ltimes \in \mathit{norm}(\phi) & \text{iff} & \mathsf{irr}(L_0) \wedge \exists L'_0 : (L_0 \preceq L'_0 \wedge L'_0.f \ltimes \in \phi) \\ \theta \in \mathit{norm}(\phi) & \text{iff} & \theta = \bigvee\{\theta' \mid \theta' \in \phi\} \end{array}$$

Similarly, we can normalize sets of abstract addresses:

Definition 7.7 *We say that a set X of abstract addresses is normalized if for all $L.f \in X$ it holds that L is irreducible.*

We write a function *norm* that converts a set of abstract addresses into an equivalent normalized set:

$$\begin{array}{rcl} \mathit{norm}(X) & = & \{x \mid x \in X\} \\ & \cup & \{L.f \mid \mathsf{irr}(L) \wedge \exists L'.f \in X \cdot L \preceq L'\} \end{array}$$

Fact 7.8 *$\mathit{norm}(X_1 \cup X_2) = \mathit{norm}(X_1) \cup \mathit{norm}(X_2)$.*

Applying *norm* gives a logically equivalent result:

Fact 7.9 *For all X, $X \blacktriangleright \mathit{norm}(X)$ and $\mathit{norm}(X) \blacktriangleright X$.*

Fact 7.10 *For all ϕ, $\phi \blacktriangleright \mathit{norm}(\phi)$ and $\mathit{norm}(\phi) \blacktriangleright \phi$.*

And applying *norm* preserves disjointness:

Fact 7.11 *$\phi \diamond X$ iff $\mathit{norm}(\phi) \diamond X$.*

Fact 7.12 *$\phi \diamond X$ iff $\phi \diamond \mathit{norm}(X)$.*

Proof: For "only if", we exploit that if $L \diamond L_2$ and $L_1 \preceq L_2$ then $L \diamond L_1$. For "if", we exploit that by Assumption 7.4 we can write any L_0 as $L_1 \sqcup \ldots \sqcup \ldots L_n$ with each $L_1 \ldots L_n$ irreducible, so if $L \diamond L_1 \ldots L \diamond L_n$ then by (the last part of) Assumption 7.3 also $L \diamond L_0$. ■

7.1 Checking Logical Implication

In Sec. 4.3, we gave a semantic definition (4.12) of logical implication. We shall show that that definition is equivalent to a syntactic characterization which is readily implementable, at least in the case where the only programmer assertion in ϕ is *true*, in which case we shall sloppily write that ϕ is "*without* programmer assertions".

Definition 7.13 *For normalized ψ and ψ', we write $\psi \leq \psi'$ iff the following holds:*

(a) *if $x \rightsquigarrow L' \in \psi'$ there exists L with $L \preceq L'$ such that $x \rightsquigarrow L \in \psi$*

(b) *if $L_1.f \rightsquigarrow L' \in \psi'$ there exists L with $L \preceq L'$ such that $L_1.f \rightsquigarrow L \in \psi$*

(c) *$x\ltimes \in \psi'$ implies $x\ltimes \in \psi$*

(d) *$L.f\ltimes \in \psi'$ implies $L.f\ltimes \in \psi$;*

(e) *$\theta' \in \psi'$ implies that there exists $\theta \in \psi$ such that $\theta \blacktriangleright \theta'$.*

For arbitrary ϕ and ϕ', we shall – with abuse of notation – write $\phi \leq \phi'$ iff $norm(\phi) \leq norm(\phi')$.

Now consider the case with no programmer assertions. Then clause (e) above is trivially true, as $\theta' = \theta =$ *true*, so it is easy to decide $\leq$. As shown by the results below, this amounts to deciding $\blacktriangleright$.

Definition 7.14 *For normalized ψ_1 and ψ_2, we define $\psi = \psi_1 \sqcup \psi_2$ as follows:*

- *$x \rightsquigarrow L \in \psi$ iff there exists L_1 and L_2 with $L = L_1 \sqcup L_2 \neq \top$ such that $x \rightsquigarrow L_1 \in \psi_1$ and $x \rightsquigarrow L_2 \in \psi_2$*
- *$L_0.f \rightsquigarrow L \in \psi$ iff there exists L_1 and L_2 with $L = L_1 \sqcup L_2 \neq \top$ such that $L_0.f \rightsquigarrow L_1 \in \psi_1$ and $L_0.f \rightsquigarrow L_2 \in \psi_2$*
- *$x\ltimes \in \psi$ iff $x\ltimes \in \psi_1$ and $x\ltimes \in \psi_2$*
- *$L.f\ltimes \in \psi$ iff $L.f\ltimes \in \psi_1$ and $L.f\ltimes \in \psi_2$*
- *$\theta \in \psi$ iff there exists $\theta_1 \in \psi_1$, $\theta_2 \in \psi_2$ such that $\theta = \theta_1 \vee \theta_2$.*

For arbitrary ϕ_1 and ϕ_2, we shall – with abuse of notation – write $\phi_1 \sqcup \phi_2$ for $norm(\phi_1) \sqcup norm(\phi_2)$.

Fact 7.15 *For all ϕ_1 and ϕ_2, $\phi_1 \sqcup \phi_2$ is normalized.*

Fact 7.16 *Given ϕ_1 and ϕ_2, $\phi_1 \sqcup \phi_2$ is their least upper bound wrt. $\leq$.*

Lemma 7.17 *If $\phi \leq \phi'$ then $\phi \blacktriangleright \phi'$.*

Proof: By Fact 7.10, it is sufficient to prove the result for normalized ϕ and ϕ'; this can be done by a straightforward case analysis on Definition 7.13. ■

Lemma 7.18 *Assume there are no programmer assertions. Then $\phi \blacktriangleright \phi'$ implies $\phi \leq \phi'$.*

To see why we need to assume the absence of programmer assertions, observe that $x = c$ logically implies $x\ltimes$ whereas $(x = c) \preceq x\ltimes$ does not hold. For that assumption to be removed, we would need a much stronger version of *norm* that finds all instances of logical implication hidden in programmer assertions.

*** THE BELOW PROOF DOES APPEAR QUITE SHAKY, BUT THE RESULT STILL SEEMS VERY PLAUSIBLE

For the proof of Lemma 7.18, due to Fact 7.10, it suffices to consider the case where ϕ and ϕ' are both normalized. We first show the following auxiliary result:

Proposition 7.19 *Assume ϕ is normalized, and contains no programmer assertions. Then there exists $s, h, s_1, h_1, \beta, \eta, \eta_1$ such that*

$$(s, h)\&(s_1, h_1) \models_{\beta,\eta,\eta_1} \phi.$$

Proof: First we define some locations:

- For each x (of pointer type) occurring in ϕ, define a location ℓ_x
- For each irreducible L_0 such that $L_0.f$ (with f of pointer type) occurs in ϕ, define a location $\ell_{L_0.f}$.
- For each irreducible L_0 occurring in ϕ, define a location ℓ_{L_0}.

Next we define s and h in the following manner.

- For each x (of pointer type) occurring in ϕ, let $s(x) = \ell_x$; if x is of integer type, let $s(x) = 0$.
- For h, we define $dom(h)$ to be all locations of the form ℓ_x, ℓ_{L_0} and $\ell_{L_0.f}$ such that:

  $h(\ell_{L_0}).f = \ell_{L_0.f}$ when f is of pointer type
  $h(\ell_{L_0}).f = 0$ when f is of integer type
  $h(\ell_x).f = \mathit{default}(\mathit{type}\, f)$.

Next, we build up η over h.

- For each x, if there exists L such that $x \rightsquigarrow L \in \phi$, then for all L', $(\ell_x)\,\eta\, L'$ iff $L \preceq L'$. In particular, note that for any x if there exists L such that $x \rightsquigarrow L \in \phi$, then $(\ell_x)\,\eta\, L$.

  For each x, if there is no L such that $x \rightsquigarrow L \in \phi$, then $(\ell_x)\,\eta\, L'$ iff $L' = \top$.
- For all irreducible locations L_0, for all L', $(\ell_{L_0})\,\eta\, L'$ iff $L_0 \preceq L'$. In particular, note that for any irreducible L_0, then $(\ell_{L_0})\,\eta\, L_0$.
- For each $L_0.f$, if there exists L such that $L_0.f \rightsquigarrow L \in \phi$, then for all L', $(\ell_{L_0.f})\,\eta\, L'$ iff $L \preceq L'$. In particular, note that for any $L_0.f$, if there exists L such that $L_0.f \rightsquigarrow L \in \phi$, then $(\ell_{L_0.f})\,\eta\, L$.

  For each $L_0.f$, if there is no L such that $L_0.f \rightsquigarrow L \in \phi$, then $(\ell_{L_0.f})\,\eta\, L'$ iff $L' = \top$.

Next, we build up β over $h\&h$.

- For each $x \ltimes \in \phi$, let $\ell_x\,\beta\,\ell_x$.
- For each irreducible L_0, we let $(\ell_{L_0})\,\beta\,(\ell_{L_0})$.
- For each $L_0.f \ltimes \in \phi$, let $(\ell_{L_0.f})\,\beta\,(\ell_{L_0.f})$.

Finally, to finish the proof, choose $s_1 = s$ and $h_1 = h$ and $\eta_1 = \eta$. It is not difficult to see that the by the construction above,

$$(s, h)\&(s_1, h_1) \models_{\beta,\eta,\eta_1} \phi.$$

■

We now return to the proof of Lemma 7.18.

Proof: Assume that $\phi \leq \phi'$ does *not* hold; we shall argue that ϕ does *not* logically imply ϕ'. To see this, use Proposition 7.19 to find $s, h, s_1, h_1, \beta, \eta, \eta_1$ such that

$$(s, h)\&(s_1, h_1) \models_{\beta,\eta,\eta_1} \phi.$$

We shall argue that

$$(s, h)\&(s_1, h_1) \models_{\beta,\eta,\eta_1} \phi'$$

does *not* hold. We do a case analysis on the reason why $\phi \leq \phi'$ does *not* hold. Thus we consider the negation of Definition 7.13.

- We have $x \rightsquigarrow L' \in \phi'$ and $L' \neq \top$; also for all L, $x \rightsquigarrow L \in \phi$ implies $L \not\preceq L'$. We first assume that $x \rightsquigarrow L \in \phi$ so that $L \not\preceq L'$. By construction, $s(x) = \ell_x$ and $(\ell_x)\,\eta\, L$. Because $L \preceq L'$ does not hold, $(\ell_x)\,\eta\, L'$ does not hold.

  Next, assume that x is such that for no L, $x \rightsquigarrow L \in \phi$. By construction, $s(x) = \ell_x$ and $(\ell_x)\,\eta\,\top$. To show $(\ell_x)\,\eta\, L'$, we need $L' = \top$ but this contradicts our assumptions. So $(\ell_x)\,\eta\, L'$ does not hold.
- We have $L_0.f \rightsquigarrow L' \in \phi'$ and $L' \neq \top$; also for all L, $L_0.f \rightsquigarrow L \in \phi$ implies $L \not\preceq L'$. We first assume that $L_0.f \rightsquigarrow L \in \phi$, so that $L \not\preceq L'$. From

$L_0.f \rightsquigarrow L \in \phi$ and
$(s,h) \models_\eta L_0.f \rightsquigarrow L$ and
$(\ell_{L_0})\,\eta\,L_0$,

we get $(h(\ell_{L_0}).f)\,\eta\,L$, i.e., $(\ell_{L_0.f})\,\eta\,L$. Because $L \preceq L'$ does not hold, $(\ell_{L_0.f})\,\eta\,L'$ does not hold. Thus although $\ell_{L_0}\,\eta\,L_0$ holds, $(h(\ell_0).f)\,\eta\,L'$ does not hold; hence $(s,h) \models_\eta L_0.f \rightsquigarrow L'$ does not hold.

Next, assume that $L_0.f$ is such that for no L, $L_0.f \rightsquigarrow L \in \phi$. Then $(\ell_{L_0.f})\,\eta\,\top$. To show $(\ell_{L_0.f})\,\eta\,L'$, we require that $L' = \top$. But this contradicts our assumptions. Hence $(\ell_{L_0.f})\,\eta\,L'$ does not hold.

- $x\ltimes \in \phi'$ but $x\ltimes \notin \phi$. Then, by construction, $\ell_x\,\beta\,\ell_x$ does not hold.
- $L_0.f\ltimes \in \phi'$ but $L_0.f\ltimes \notin \phi$. Then $(\ell_{L_0})\,\eta\,L_0$ and $(\ell_{L_0})\beta(\ell_{L_0})$ and $h(\ell_{L_0}).f = \ell_{L_0.f}$, so if $(s,h)\&(s_1,h_1) \models_{\beta,\eta,\eta_1} L_0.f\ltimes$, then $(\ell_{L_0.f})\,\beta\,(\ell_{L_0.f})$, which by construction of β, does not hold.

■

By combining Lemmas 7.17 and 7.18, we get:

Theorem 7.20 *If ϕ and ϕ' contains no programmer assertions, then $\phi \blacktriangleright \phi'$ is equivalent to $\phi \leq \phi'$.*

Concerning how to decide $X \blacktriangleright X'$, we have a similar (but simpler) result.

Fact 7.21 *If $X \blacktriangleright X'$ then $norm(X) \subseteq norm(X')$.*

Proof: Let $L.f \in norm(X)$ (the other case is similar). We can find s, h, s', h' such that $\ell.f$ is modified from h to h' and with $\ell\,\eta\,L$ we thus have

$$(s,h) \rightarrow (s,h') \models_\eta norm(X)$$

which by Fact 7.9 amounts to

$$(s,h) \rightarrow (s,h') \models_\eta X$$

which, since $X \blacktriangleright X'$, implies

$$(s,h) \rightarrow (s,h') \models_\eta X'$$

which by Fact 7.9 amounts to

$$(s,h) \rightarrow (s,h') \models_\eta norm(X')$$

so there exists L' with $\ell\,\eta\,L'$ such that $L'.f \in norm(X')$. By assumption, L and L' are irreducible. But $L \diamond L'$ does not hold, so by assumption, we infer $L = L'$. Thus, $L.f \in norm(X')$, as desired. ■

Lemma 7.22 *$X \blacktriangleright X'$ iff $norm(X) \subseteq norm(X')$.*

Proof: The "only if" direction was provided by Fact 7.21. For the "if" direction, Fact 4.13 tells us that $norm(X) \blacktriangleright norm(X')$; and by Fact 7.9 therefore $X \blacktriangleright X'$. ■

Lemma 7.23 *If $X_1 \blacktriangleright X$ and $X_2 \blacktriangleright X$ then $X_1 \cup X_2 \blacktriangleright X$.*

Proof: from previous results we have

$$\begin{array}{rcl} norm(X_1 \cup X_2) & = & norm(X_1) \cup norm(X_2) \\ & \subseteq & norm(X) \cup norm(X) \\ & = & norm(X) \end{array}$$

and the claim now follows from Lemma 7.22. ■

Lemma 7.24 *If $X \blacktriangleright X'$ and $a \diamond X'$ then $a \diamond X$.*

Proof: Assume, to get a contradiction, that $a \diamond X$ does not hold. There are two cases.

<u>a is a variable x.</u> Then $x \in X$. It is clearly possible to find s, h, s', h' such that x is modified from s to s', but no other concrete address is modified from h to h' or from s to s'. Thus $(s, h) \rightarrow (s', h') \models_\eta X$. So since X logically implies X', $(s, h) \rightarrow (s', h') \models_\eta X'$. Therefore $x \in X'$, contradicting $a \diamond X'$.

<u>a is of the form $L.f$.</u> Then there exists $L_1.f \in X$ such that $L \diamond L_1$ does not hold. Then it is possible to define η such that $\ell\,\eta\,L$ and $\ell\,\eta\,L_1$. It is also possible to find s, h, s', h' such that $\ell.f$ is modified from h to h', but no other concrete address is modified from h to h' or from s to s'. Thus $(s, h) \rightarrow (s', h') \models_\eta X$. So since X logically implies X', $(s, h) \rightarrow (s', h') \models_\eta X'$ holds also. That is, there exists $L_2.f \in X'$ with $\ell\,\eta\,L_2$. But then $L \diamond L_2$ does not hold, and therefore $a \diamond X'$ does not hold, yielding the desired contradiction.
■

Using Lemma 7.24, we easily get:

Lemma 7.25 *if $X \blacktriangleright X'$ and $\phi \diamond X'$ then $\phi \diamond X$.*

7.2 A Sound Algorithm

We shall define, inductively on S, a function $sp(S, \phi_0)$ that given a command S and a precondition ϕ_0 (which could be "global") computes a pair (ϕ, X); here we want ϕ to be a postcondition of S, and X to be the abstract addresses that may be modified by S. With

Assumption 7.26 *We assume that a consistent summary environment Π is given in advance*

we can show soundness of sp wrt. to the logic:

Theorem 7.27 *If $sp(S, \phi_0) = (\phi, X)$ then $\Pi \vdash \{\phi_0\}\ S\ \{\phi\}\ [X]$.*

Below we shall present sp, proving its soundness along the way.

<u>Case of sequential composition.</u> Define

$sp(S_1\,; S_2, \phi_0) =$
  let $(\phi_1, X_1) = sp(S_1, \phi_0)$ in
  let $(\phi, X_2) = sp(S_2, \phi_1)$ in
 $(\phi, X_1 \cup X_2)$

Proof of soundness: Inductively, we have

$\{\phi_0\}\ S_1\ \{\phi_1\}\ [X_1]$ and
$\{\phi_1\}\ S_2\ \{\phi\}\ [X_2]$

and therefore the desired

$\{\phi_0\}\ S_1\,; S_2\ \{\phi\}\ [X_1 \cup X_2]$

<u>Case of conditionals.</u>

We call sp recursively on the two branches and then combine, via the least upper bound operator, the resulting assertions. Let ϕ_{12} be the least upper bound of the analyses of the branches. Looking at the side conditions for [If] in the logic, we see that if ϕ_0 logically implies $x\ltimes$ (with x the test), we can just return ϕ_{12}. Otherwise, in order to satisfy the second side condition, we must remove from ϕ_{12} all agreement assertions which either are not in the precondition, or whose abstract addresses have been modified in S_1 or in S_2. The resulting code is

$sp(\textbf{if } x \textbf{ then } S_1 \textbf{ else } S_2, \phi_0) =$
$\quad$ let $(\phi_1, X_1) = sp(S_1, \phi_0)$ in
$\quad$ let $(\phi_2, X_2) = sp(S_2, \phi_0)$ in
$\quad$ let $X = norm(X_1 \cup X_2)$ in
$\quad$ let $\phi_{12} = \phi_1 \sqcup \phi_2$ in
$\quad$ let $\phi =$ if $\phi_0 \leq x\ltimes$
$\quad\quad$ then ϕ_{12}
$\quad\quad$ else $\phi_{12} \setminus (C_1 \cup C_2)$
$\quad$ where $C_1 = \{y\ltimes \mid (y \in X) \vee (y\ltimes \notin norm(\phi_0))\}$
$\quad$ and $C_2 = \{L.f\ltimes \mid (L.f \in X) \vee (L.f \notin norm(\phi_0))\}$
in (ϕ, X)

Proof of soundness: Inductively, we have

$\{\phi_0\}\ S_1\ \{\phi_1\}\ [X_1]$
$\{\phi_0\}\ S_2\ \{\phi_2\}\ [X_2]$

and by [Conseq] then

$\{\phi_0\}\ S_1\ \{\phi\}\ [X]$
$\{\phi_0\}\ S_2\ \{\phi\}\ [X]$

which implies the desired judgement

$\{\phi_0\}\ \textbf{if } x \textbf{ then } S_1 \textbf{ else } S_2\ \{\phi\}\ [X]$

since if $\phi_0 \blacktriangleright x\ltimes$ does not hold then (by Lemma 7.17) neither does $\phi_0 \leq x\ltimes$ so by construction of ϕ:

$\mathcal{I}(\phi) \diamond X$ and
$\phi_0 \blacktriangleright \mathcal{I}(\phi)$

Case for while.

```
*** TO BE WRITTEN ***
Involves fixed point iteration
```

Case for assertions.

$$sp(\textbf{assert } \theta, \phi_0) = (\theta \wedge \phi_0, \emptyset)$$

Proof of soundness: By logic,

$\{true\}\ \textbf{assert } \theta\ \{\theta\}\ [\emptyset]$

and by the frame rule therefore (since clearly $\phi_0 \diamond \emptyset$)

$\{\phi_0\}\ \textbf{assert } \theta\ \{\theta \wedge \phi_0\}\ [\emptyset]$

Case of assignments.

Assume that S is an assignment A which is not a method call, i.e., A is either a pure assignment, a pointer assignment, a null assignment, a field access, a field update, or an object creation. Assume that we have a nondeterministic function $Choose(A, \phi_0)$ which returns a triple (ψ_0, ψ, X) such that $\{\psi_0\}\ A\ \{\psi\}\ [X]$ is an instance of a rule for A in the logic where $\phi_0 \leq \psi_0$ (and hence $\phi_0 \blacktriangleright \psi_0$). Then define

$sp(S, \phi_0) =$
$\quad$ let $(\psi_0, \psi, X) = Choose(A, \phi_0)$ in
$\quad$ let $\phi = \psi \wedge disj(\phi_0, X)$ in (ϕ, X)

Here, the function $disj$ extracts the parts of an assertion *not modified* by the assignment, thus incorporating the frame rule. It is defined by $disj(\phi, X) = \{\alpha \in norm(\phi) \mid \alpha \diamond X\}$.

So far, the above definition is very non-deterministic; it will be concretized in the next section when we consider *strongest* postconditions.

Proof of soundness: By the definition of $Choose$, there is a rule in the logic with instance

$$\{\psi_0\}\ A\ \{\psi\}\ [X]$$

from which we by [Frame] infer

$$\{\psi_0 \wedge disj(\phi_0, X)\}\ A\ \{\psi \wedge disj(\phi_0, X)\}\ [X]$$

and by [Conseq] the desired judgement

$$\{\phi_0\}\ A\ \{\phi\}\ [X].$$

Case for method calls.

Assume that S is a method call $x := y.m(w)$, with $type\, y = C$ where C contains the method m with formal parameter z. Assume that we have a non-deterministic function[16] $Choose(m, C, \phi_0)$ which returns a triple (ψ_0, ψ, X) such that $\{\psi_0\}\ \_\ \{\psi\}\ [X] \in \Pi(C, m)$ where $\phi_0 \leq \psi_0[y/self, w/z]$. Then:

$$\begin{array}{l} sp(S, \phi_0) = \\ \quad \mathsf{let}\ (\psi_0, \psi, X) = Choose(m, C, \phi_0)\ \mathsf{in} \\ \quad \mathsf{let}\ \phi_X = disj(\phi_0, X \cup \{x\})\ \mathsf{in} \\ \quad \mathsf{let}\ \phi = \psi[x/result] \cup \phi_X\ \mathsf{in}\ (\phi, X \cup \{x\}) \end{array}$$

Proof of soundness: By the definition of $Choose$, an application of [MethodCall] enables us to infer

$$\{\psi_0[y/self, w/z]\}\ x := y.m(w)\ \{\psi[x/result]\}\ [X \cup \{x\}]$$

from which we by [Frame] infer

$$\begin{array}{l} \{\psi_0[y/self, w/z] \wedge \phi_X\} \\ \quad x := y.m(w) \\ \{\psi[x/result] \wedge \phi_X\} \quad [X \cup \{x\}] \end{array}$$

and by [Conseq] the desired judgement

$$\{\phi_0\}\ x := y.m(w)\ \{\phi\}\ [X \cup \{x\}].$$

Construction of method summaries. In an actual implementation, the summary environment Π may be built incrementally, by using sp to analyze a new method in the context of the current Π (see, e.g., [24]). For recursive methods, however, the user might be required to provide the summaries, as in ESC/Java [15].

7.3 Strongest Postcondition

We shall now look at conditions for when sp, as defined in the previous section, is indeed the *strongest* postcondition. For that purpose, we need to control the nondeterminism in the selection of abstract locations in rule [New].

Assumption 7.28 *Each occurrence of "new" is associated with a specific irreducible abstract location L_0 such that the only rule applicable for that occurrence is*

$$\{true\}\ x := \mathbf{new}\ C\ \{x \rightsquigarrow L_0;\ x\ltimes\}\ [\{x\}].$$

[16]Required because we have a set of summaries for different calling contexts, so we need to select the appropriate one.

$Choose(x := \mathbf{new}\ C, \phi_0) =$
  let L_0 be the designated abstract location
    for this occurrence of "new" in
  $\{x \rightsquigarrow L_0, x\ltimes\}, \{x\})$

$Choose(x := z, \phi_0) =$
  let $L = \phi_0(z)$ in
  if $\phi_0 \leq z\ltimes$
  then $(\{z \rightsquigarrow L, z\ltimes\}, \{x \rightsquigarrow L, x\ltimes\}, \{x\})$
  else $(\{z \rightsquigarrow L\}, \{x \rightsquigarrow L\}, \{x\})$

$Choose(x := y.f, \phi_0) =$
  let $L = \phi_0(y) = L_1 \sqcup \ldots \sqcup L_k$ in
  let $LI = \sqcup_{j \in 1 \ldots k} \phi_0(L_j.f)$ in
  if $\phi_0 \leq y\ltimes, L.f\ltimes$
  then $(\{y \rightsquigarrow L, L.f \rightsquigarrow LI, y\ltimes, L.f\ltimes\}, \{x \rightsquigarrow LI, x\ltimes\}, \{x\})$
  else $(\{y \rightsquigarrow L, L.f \rightsquigarrow LI\}, \{x \rightsquigarrow LI\}, \{x\})$

$Choose(x := E, \phi_0) =$
  let $z_1, \ldots, z_n = \mathrm{free}(E)$ in
  if $\phi_0 \leq z_1\ltimes, \ldots, z_n\ltimes$
  then $(\{z_1\ltimes, \ldots, z_n\ltimes\}, \{x\ltimes\}, \{x\})$
  else $(\{\}, \{\}, \{x\})$

$Choose(x := \mathbf{null}, \phi_0) = (\{\}, \{x \rightsquigarrow \bot, x\ltimes\}, \{x\})$

$Choose(x.f := y, \phi_0) =$
  let $L = \phi_0(x)$ in
  let $LI' = \phi_0(y)$ in
  let $L = L_1 \sqcup \ldots \sqcup L_k$ in (irreducible)
  let $LI = \sqcup_{j \in \{1 \ldots k\}} \phi_0(L_j.f) \sqcup LI'$ in
  if $\phi_0 \leq x\ltimes, y\ltimes, L.f\ltimes$
  then $(\{x \rightsquigarrow L, y \rightsquigarrow LI, L.f \rightsquigarrow LI, x\ltimes, y\ltimes, L.f\ltimes\},$
    $\{L.f \rightsquigarrow LI, L.f\ltimes\}, \{L.f\})$
  else $(\{x \rightsquigarrow L, y \rightsquigarrow LI, L.f \rightsquigarrow LI\}, \{L.f \rightsquigarrow LI\}, \{L.f\})$

Table 3: The function *Choose*, given normalized ϕ_0.

Then we can concretize, as done in Table 3, the function *Choose* for assignments. Thanks to Assumption 7.28, we can show that *Choose* computes the "strongest applicable version".

Definition 7.29 (Strongest Applicable Version) *Given rule schema ($j \in J$), $\{\psi_j\}$ S $\{\psi'_j\}$ $[X_j]$. For given ϕ_0, we say that j_0 is the strongest applicable version if*

- $\phi_0 \leq \psi_{j_0}$
- *For all j such that $\phi_0 \leq \psi_j$, it holds that $\psi'_{j_0} \leq \psi'_j$ and $X_{j_0} \blacktriangleright X_j$.*

Lemma 7.30 *For all kinds of assignments, Choose as given in Table 3 computes the strongest applicable version.*

Proof: Rather straightforward, and relatively easy, except for the case for field update which is somewhat tedious. ■

Assumption 7.31 *The method summaries have been constructed such that there exists a strongest applicable version for method calls.*

Before embarking on the completeness theorem and its proof, we must be able to control the application of non-structural rules.

7.3.1 Normalized derivations

Fact 7.32 *If from $\{\psi_0\}$ S $\{\psi\}$ $[X_0]$ we can arrive at $\{\phi_0\}$ S $\{\phi\}$ $[X]$ by first applying [Conseq] and next [Frame], we could alternatively arrive at the same conclusion by* first *applying [Frame] and* then *[Conseq].*

Proof: Our assumptions are

$$\frac{\{\psi_0\}\ S\ \{\psi\}\ [X_0]}{\{\phi_0'\}\ S\ \{\phi'\}\ [X_0]}\quad[\mathsf{Conseq}]$$
$$\text{and}\ \frac{\{\phi_0'\}\ S\ \{\phi'\}\ [X_0]}{\{\phi_0\}\ S\ \{\phi\}\ [X]}\quad[\mathsf{Frame}]$$

where there exists ϕ_1 with $\phi_1 \diamond X_0$ such that

$\phi_0 = \phi_0' \wedge \phi_1$,
$\phi = \phi' \wedge \phi_1$

and where ϕ_0' logically implies ψ_0 and ψ logically implies ϕ' and X_0 logically implies X.

But then, since $\phi_1 \diamond X_0$ holds by Lemma 7.25, we also have the derivation

$$\frac{\{\psi_0\}\ S\ \{\psi\}\ [X_0]}{\{\psi_0 \wedge \phi_1\}\ S\ \{\psi \wedge \phi_1\}\ [X_0]}\quad[\mathsf{Frame}]$$
$$\text{and}\ \frac{\{\psi_0 \wedge \phi_1\}\ S\ \{\psi \wedge \phi_1\}\ [X_0]}{\{\phi_0\}\ S\ \{\phi\}\ [X]}\quad[\mathsf{Conseq}]$$

■

Fact 7.33 *If from $\{\psi_0\}\ S\ \{\psi\}\ [X_0]$ we can arrive at $\{\phi_0\}\ S\ \{\phi\}\ [X]$ by zero or more application of [Frame], we could alternatively arrive at the same conclusion by applying the frame rule exactly once.*

Proof: Trivial. ■

Fact 7.34 *If from $\{\psi_0\}\ S\ \{\psi\}\ [X_0]$ we can arrive at $\{\phi_0\}\ S\ \{\phi\}\ [X]$ by zero or more application of [Conseq], we could alternatively arrive at the same conclusion by applying [Conseq] exactly once.*

Proof: Trivial. ■

From the previous facts, we clearly have:

Lemma 7.35 *Assume that we in our logic can derive $\{\phi_0\}\ S\ \{\phi\}\ [X]$ Then we can construct a derivation of the form*

$$\frac{D_1}{D_2}\qquad \textit{syntax-directed rule for } S$$

$$\frac{D_2}{D_3}\qquad [\mathsf{Frame}]$$

$$\frac{D_3}{\{\phi_0\}\ S\ \{\phi\}\ [X]}\qquad [\mathsf{Conseq}]$$

Lemma 7.36 *Assume that in our logic we can derive*

$\{\phi_0\}\ S_1\ ;S_2\ \{\phi\}\ [X]$

Then there exists ϕ_1, X_1, X_2 such that

$\{\phi_0\}\ S_1\ \{\phi_1\}\ [X_1]$ *and*
$\{\phi_1\}\ S_2\ \{\phi\}\ [X_2]$ *and*
$X = X_1 \cup X_2$.

Proof: We know from Lemma 7.35 that there exists a derivation

$$\frac{\{\psi_0\}\ S_1\ \{\psi_1\}\ [X_1] \qquad \{\psi_1\}\ S_2\ \{\psi\}\ [X_2]}{\{\psi_0\}\ S_1\,;S_2\ \{\psi\}\ [X']}\ \textsf{[Seq]}$$

$$\frac{\{\psi_0\}\ S_1\,;S_2\ \{\psi\}\ [X']}{\{\phi'_0\}\ S_1\,;S_2\ \{\phi'\}\ [X']}\ \textsf{[Frame]}$$

$$\frac{\{\phi'_0\}\ S_1\,;S_2\ \{\phi'\}\ [X']}{\{\phi_0\}\ S_1\,;S_2\ \{\phi\}\ [X]}\ \textsf{[Conseq]}$$

where $X' = X_1 \cup X_2$ and where there exists ϕ'_1 such that

$\phi'_0 = \psi_0 \wedge \phi'_1$,
$\phi' = \psi \wedge \phi'_1$
$\phi'_1 \diamond X'$

and where

ϕ_0 logically implies ϕ'_0
ϕ' logically implies ϕ and
X' logically implies X

Clearly, $\phi'_1 \diamond X_1$ and $\phi'_1 \diamond X_2$, so we can construct derivations

$\{\psi_0 \wedge \phi'_1\}\ S_1\ \{\psi_1 \wedge \phi'_1\}\ [X_1]$
$\{\psi_1 \wedge \phi'_1\}\ S_2\ \{\psi \wedge \phi'_1\}\ [X_2]$

But since

ϕ_0 logically implies $\psi_0 \wedge \phi'_1$
$\psi \wedge \phi'_1$ logically implies ϕ
X_1 logically implies X
X_2 logically implies X
$\phi_1 = \psi_1 \wedge \phi'_1$

we have derivations of the desired form

$\{\phi_0\}\ S_1\ \{\phi_1\}\ [X]$ and
$\{\phi_1\}\ S_2\ \{\phi\}\ [X]$

■

7.3.2 Proof of completeness

Theorem 7.37 *If $sp(S, \phi_0) = (\phi, X)$ and $\{\phi_0\}\ S\ \{\phi'\}\ X'$ then $\phi \blacktriangleright \phi'$ and $X \blacktriangleright X'$.*

Proof: Go by induction on S, using that $\leq$ equals $\blacktriangleright$.

Case of assignments.

Let S be an assignment A. Remember that sp is defined by

$sp(A, \phi_0) =$
let $(\psi_0, \psi, X) = Choose(A, \phi_0)$ in
let $\phi = \psi \wedge disj(\phi_0, X)$ in
(ϕ, X)

where $disj(\phi_0, X) = \{\alpha \in norm(\phi_0) \mid \alpha \diamond X\}$.

Now assume that $\{\phi_0\}\ A\ \{\phi'\}\ [X']$ with a derivation which (due to Lemma 7.35) is of the form

$$\frac{\cdots}{\{\psi_0'\}\ A\ \{\psi'\}\ [X'']} \qquad \text{(structural rule for } A\text{)}$$

$$\frac{\{\psi_0'\}\ A\ \{\psi'\}\ [X'']}{\{\psi_0' \wedge \phi_1\}\ A\ \{\psi' \wedge \phi_1\}\ [X'']} \qquad \textsf{[Frame]}$$

$$\frac{\{\psi_0' \wedge \phi_1\}\ A\ \{\psi' \wedge \phi_1\}\ [X'']}{\{\phi_0\}\ A\ \{\phi'\}\ [X']} \qquad \textsf{[Conseq]}$$

where

ϕ_0 logically implies $\psi_0' \wedge \phi_1$
$\psi' \wedge \phi_1$ logically implies ϕ'
X'' logically implies X'
$\phi_1 \diamond X''$

Since, by Lemma 7.30, *Choose* computes the Strongest Applicable version, and since ϕ_0 logically implies ψ_0', we can conclude that

ψ logically implies ψ'
X logically implies X''

and thus X logically implies X'.

We are done if we can prove

$$\mathit{disj}(\phi_0, X) \le \phi_1 \qquad (1)$$

for then we have the desired relation

$$\phi = \psi \wedge \mathit{disj}(\phi_0, X) \le \psi' \wedge \phi_1 \le \phi'.$$

So let us embark on proving (1), where we wlog. can assume that ϕ_1 is normalized. Looking at Definition 7.13, we consider the case (the others are similar) where $L.f \rightsquigarrow L_1 \in \phi_1$. Since $\phi_1 \diamond X''$, we know that $L.f \diamond X''$, and from $X \blacktriangleright X''$ we can infer $L.f \diamond X$. Since $\phi_0 \blacktriangleright \phi_1$ and hence $\phi_0 \le \phi_1$, we know that there exists L_0 with $L_0 \preceq L_1$ such that $L.f \rightsquigarrow L_0 \in \mathit{norm}(\phi_0)$. Thus $L.f \rightsquigarrow L_0 \in \{\alpha \in \mathit{norm}(\phi_0) \mid \alpha \diamond X\} = \mathit{disj}(\phi_0, X)$, establishing (1).

Case of sequential composition.

Here sp is given by

$$\begin{array}{l} sp(S_1\,;S_2, \phi_0) = \\ \mathsf{let}\ (\phi_1, X_1) = sp(S_1, \phi_0)\ \mathsf{in} \\ \mathsf{let}\ (\phi, X_2) = sp(S_2, \phi_1)\ \mathsf{in} \\ (\phi, X_1 \cup X_2) \end{array}$$

Assume that $\{\phi_0\}\ S_1\,;S_2\ \{\phi'\}\ [X']$. By Lemma 7.36, there exists ϕ_1', X_1', X_2' with $X' = X_1' \cup X_2'$ such that

$\{\phi_0\}\ S_1\ \{\phi_1'\}\ [X_1']$ and
$\{\phi_1'\}\ S_2\ \{\phi'\}\ [X_2']$

Inductively on S_1, we have

ϕ_1 logically implies ϕ_1' and
X_1 logically implies X_1'

implying that we have the judgement

$\{\phi_1\}\ S_2\ \{\phi'\}\ [X'_2].$

Inductively on S_2, we now have

ϕ logically implies ϕ' and
X_2 logically implies X'_2.

The result now follows from Lemma 7.22, observing that

$$\begin{array}{rcl} norm(X_1 \cup X_2) & = & norm(X_1) \cup norm(X_2) \\ & \subseteq & norm(X'_1) \cup norm(X'_2) \\ & = & norm(X') \end{array}$$

and therefore $X_1 \cup X_2$ logically implies X'.

Case of conditionals.

Here sp is defined by

$sp(\textbf{if } x \textbf{ then } S_1 \textbf{ else } S_2, \phi_0) =$
let $(\phi_1, X_1) = sp(S_1, \phi_0)$ in
let $(\phi_2, X_2) = sp(S_2, \phi_0)$ in
let $X = norm(X_1 \cup X_2)$ in
let $\phi_{12} = \phi_1 \sqcup \phi_2$ in
let $\phi =$
if ϕ_0 logically implies $x\ltimes$
then ϕ_{12}
else $\phi_{12} - (C_1 \cup C_2)$
where $C_1 = \{y\ltimes \mid (y \in X) \vee (y\ltimes \notin norm(\phi_0))\}$
and $C_2 = \{L.f\ltimes \mid (L.f \in X) \vee (L.f \notin norm(\phi_0))\}$
in (ϕ, X)

Now assume that $\{\phi_0\}$ **if** x **then** S_1 **else** S_2 $\{\phi'\}$ $[X']$ and by Lemma 7.35 we can assume that we have a derivation

$$\frac{\{\psi_0\}\ S_1\ \{\psi'\}\ [X'']\quad \{\psi_0\}\ S_2\ \{\psi'\}\ [X'']}{\{\psi_0\}\ \textbf{if}\ x\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \{\psi'\}\ [X'']}\ [\mathsf{If}]$$

$$\frac{\{\psi_0\}\ \textbf{if}\ldots\ \{\psi'\}\ [X'']}{\{\psi_0 \wedge \psi\}\ \textbf{if}\ldots\ \{\psi' \wedge \psi\}\ [X'']}\ [\mathsf{Frame}]$$

$$\frac{\{\psi_0 \wedge \psi\}\ \textbf{if}\ldots\ \{\psi' \wedge \psi\}\ [X'']}{\{\phi_0\}\ \textbf{if}\ldots\ \{\phi'\}\ [X']}\ [\mathsf{Conseq}]$$

where

$\phi_0 \leq (\psi_0 \wedge \psi)$
$(\psi' \wedge \psi) \leq \phi'$
X'' logically implies X'
$\psi \diamond X''$

and if $\psi_0 \leq x\ltimes$ does not hold then $\mathcal{I}(\psi') \diamond X''$ and $\psi_0 \leq \mathcal{I}(\psi')$. By applying the frame rule and the subsumption rule we get

$\{\phi_0\}\ S_1\ \{\psi' \wedge \psi\}\ [X'']$
$\{\phi_0\}\ S_2\ \{\psi' \wedge \psi\}\ [X'']$

so by the induction hypothesis we have

$\phi_1 \leq (\psi' \wedge \psi)$
$\phi_2 \leq (\psi' \wedge \psi)$
X_1 logically implies X''
X_2 logically implies X''

from which we infer

$\phi_{12} \leq (\psi' \wedge \psi)$

and thus $\phi_{12} \leq \phi'$.

Also, by Lemma 7.22 we see that X logically implies X'' and thus X logically implies X'.

If ϕ_0 logically implies $x\ltimes$, then $\phi = \phi_{12}$, so $\phi \leq \phi'$ follows and we are done.

So now assume that ϕ_0 does not logically imply $x\ltimes$. But then ψ_0 does not logically imply $x\ltimes$, so it holds that

$\psi_0 \leq \mathcal{I}(\psi')$ and
$\mathcal{I}(\psi') \diamond X''$

Our task is to prove that

$\phi \leq (\psi' \wedge \psi)$

so let $\alpha \in norm(\psi' \wedge \psi)$.

From $\phi_{12} \leq (\psi' \wedge \psi)$ we deduce that

$norm(\phi_{12}) \leq \alpha$, that is,
$\phi_{12} \leq \alpha$.

This will imply also $\phi \leq \alpha$, except for the case where α takes the form $a\ltimes$ (and thus $\alpha \in \phi_{12}$).

So let us consider the case where α is of the form $y\ltimes$ (the case $\alpha = L.f\ltimes$ is similar). From $y\ltimes \in norm(\psi' \wedge \psi)$ we deduce, *using that norm does not have other rules than listed* that

$$y\ltimes \in norm(\mathcal{I}(\psi') \wedge \psi)) \tag{2}$$

Observe that, since X logically implies X'', by Lemma 7.25 we have

$\mathcal{I}(\psi') \diamond X$ and
$\psi \diamond X$

From Fact 7.11 we thus have

$norm(\mathcal{I}(\psi') \wedge \psi) \diamond X$

showing that

$$y \notin X \tag{3}$$

From the above we also see that ϕ_0 logically implies $\psi_0 \wedge \psi$ which logically implies $\mathcal{I}(\psi') \wedge \psi$, that is $\phi_0 \leq (\mathcal{I}(\psi') \wedge \psi)$. So from (2) we infer that

$$y\ltimes \in norm(\phi_0) \tag{4}$$

But from (3) and (4), and $y\ltimes \in \phi_{12}$, we infer that $y\ltimes \in \phi$, as desired.

Case of assertions.

Here sp is defined by

$$sp(\textbf{assert }(\theta), \phi_0) = (\theta \wedge \phi_0, \emptyset)$$

Now assume that $\{\phi_0\}$ **assert** θ $\{\phi'\}$ $[X']$ because of a derivation which by Lemma 7.35 is of form

$$\frac{\{true\}\ \textbf{assert}\ \theta\ \{\theta\}\ [\emptyset]}{\{\phi_1\}\ \textbf{assert}\ \theta\ \{\theta \wedge \phi_1\}\ [\emptyset]}\quad [\mathsf{Frame}]$$

$$\frac{\{\phi_1\}\ \textbf{assert}\ \theta\ \{\theta \wedge \phi_1\}\ [\emptyset]}{\{\phi_0\}\ \textbf{assert}\ \theta\ \{\phi'\}\ [X'']}\quad [\mathsf{Conseq}]$$

where ϕ_0 logically implies ϕ_1, and $\theta \wedge \phi_1$ logically implies ϕ'. But then $\theta \wedge \phi_0$ logically implies ϕ', as desired.

Case of method calls. This is very similar to the case for assignments, using Assumption 7.31 rather than Lemma 7.30. ■

8 Discussion

We have specified, via a Hoare-style logic, an interprocedural and flow-sensitive information flow analysis for object-oriented programs. (The analysis is *in*sensitive to termination, but we expect that adding assertions of the form $\perp\ltimes$, c.f. [2], would make it sensitive to termination). Because aliasing can compromise confidentiality, the logic uses points-to assertions to describe aliasing that may arise between variables and between heap values. Agreement assertions describe the absence of leaks due to data and control flow in a program. Together with the knowledge that particular abstract addresses are disjoint, i.e., they must not alias, the logic can be employed to specify a more precise information flow analysis than extant type-based approaches. We also permit JML style programmer assertions in code. Such assertions allow more programs to be deemed secure than would be permitted by points-to and agreement assertions alone, albeit at the cost of a fully automatic checker.

Local reasoning about state is supported in our logic and we show a number of examples. While ordinary Hoare logic without aliasing is compositional by nature, aliasing makes it challenging to reason locally about the heap. By drawing upon fundamental ideas from separation logic, we achieve local reasoning: we use small specifications for each command and combine specifications via a frame rule. The small specifications only mention abstract addresses relevant to a command and semantically correspond to the footprint of the command in the global state [20]. The frame rule permits a move from local to non-local specifications.

As we mentioned in Sec. 5, Table 2 specifies two sets of rules. The reader might have noted that the rules that mention *points-to assertions only* specify a points-to analysis similar to well-known ones, e.g., [12, 8]. Data flow facts used in typical points-to analyses can be viewed as assertions. Nevertheless, we have not found in the literature an explicit Hoare-style specification of interprocedural points-to analysis that is based on local reasoning via small specifications and the frame rule. On top of such a points-to analysis, a host of other analyses (rather than just information flow analysis) could be specified.

There is much work that remains. We wish to experimentally validate whether local reasoning with the frame rule indeed provides scalability. Towards this goal, we plan to extend ESC/Java2[17] and its assertion language, JML [11], to handle points-to and agreement assertions. This would provide a verification framework for information flow properties. For checking benchmarks (e.g.,[4]) that use declassification, we conjecture that agreement assertions might help in statically predicting program points where declassification may be used.

[17] `http://secure.ucd.ie/products/opensource/ESCJava2`

A significantly harder problem is obtaining a modular interprocedural *analysis*. This requires devising a modular algorithm for computing strongest postconditions, one that discovers and updates procedure summaries on the fly. We plan to explore how local reasoning might be employed in this process.

Although our logic does not have separation logic's spatial conjunction ($\star$) operator, we conjecture that the semantics of assertions could be alternatively given as follows: the meaning of e.g., $x \rightsquigarrow L$ in state (s, h) under η, could consider a partition of h into disjoint subheaps h_1, h_2 such that $dom(h_1) = \{s(x)\}$ with $(s(x))\,\eta\,L$.

Our hope is that local reasoning will be used in the specification of program analyses and — in the security context — used as a foundation for checking security policies for practical systems composed of components.

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