Visibility and Separability for a Declarative Linearizability Proof of the Timestamped Stack

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Abstract

Linearizability is a standard correctness criterion for concurrent algorithms, typically proved by establishing the algorithms' linearization points (LP). However, LPs often hinder abstraction, and for some algorithms such as the timestamped stack, it is unclear how to even identify their LPs. In this paper, we show how to develop declarative proofs of linearizability by foregoing LPs and instead employing axiomatization of so-called visibility relations. While visibility relations have been considered before for the timestamped stack, our study is the first to show how to derive the axiomatization systematically and intuitively from the sequential specification of the stack. In addition to the visibility relation, a novel *separability* relation emerges to generalize real-time precedence of procedure invocation. The visibility and separability relations have natural definitions for the timestamped stack, and enable a novel proof that reduces the algorithm to a simplified form where the timestamps are generated atomically.

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1 Introduction

A concurrent data structure is *linearizable* [11] if in every concurrent execution history of the structure's exportable methods, the method invocations can be ordered linearly just by permuting *overlapping* invocations, so that the obtained history is *sequentially sound*; that is, executing the methods sequentially in the linear order produces the same outputs that the methods had in the concurrent history. In other words, every concurrent history is equivalent to a sequential one where methods execute without interference, i.e., *atomically*.

While linearizability is a standard correctness criterion, proving that sophisticated data structures are linearizable is far from trivial. The most common approach is to first describe the linearization points (LPs) of the methods that the data structure exports. Given an execution of a method (henceforth, event), its LP is the moment at which the event's effect can be considered to have occurred abstractly, in the sense that the linearization order of the events is determined by the real-time order of the chosen LPs. LPs are described operationally by indicating the line in the code together with a run-time condition under which the line applies. The proof then proceeds by a simulation argument to show that the effect of the invocation abstractly occurs at the declared line.

While LPs lead to a complete proof method [16], the operational nature of the LP description leads to very low-level proofs. Sometimes, it may even be unclear how to describe the position of the LPs in the first place. An alternative, more *declarative* approach, that offers higher levels of abstraction, has been proposed by Henzinger et al. [10]. It advocates foregoing LPs in favor of axiomatizing how the events of the structure depend on each other. Such dependence relation has since been termed *visibility relation* in the literature [17], and has been used to axiomatize concurrent queues [10], stacks [3, 7], and snapshot algorithms [14]. In these cases, the higher abstraction capabilities of visibility relations (compared to LPs) enabled that linearizability proofs of different implementations of a data structure can share significant proof components, or that a linearizability proof can be developed in the first place where an LP-based proof did not exist. Nevertheless, despite these recent successes, developing visibility-based proofs remains an undeveloped area, with every proof approaching the axiomatization in its own manner, without any specific systematization.

This paper advances the visibility approach by proposing that the axiomatization of concurrent structures should rely on a *separability* relation between events, in addition to the visibility relation. Separability relation partially characterizes when two events are *abstractly* non-overlapping, with one event logically preceding the other. Thus, it is the abstract counterpart to the "returns-before" relation, which is standard in the literature, and holds between two events if, in real time, the first event terminates before the second begins.

We employ the visibility and separability relations in tandem to derive a new axiomatization and linearizability proof for the concurrent structure of the *timestamped stack*, initially designed and proved linearizable by Dodds et al. [3] and Haas [7]. Since its inception, the timestamped stack has achieved some notoriety for the difficulty of its linearizability proof, as it has so far resisted an operational description of its LPs, and simulation-based proof attempts. For example, Khyzha et al. [12] verified the timestamped *queue* by a simulation-based approach, but did not scale to the stack. Bouajjani et al. [1] employed forward simulation on a simplified variant of the stack where timestamps are allocated atomically, but did not attempt the general variant, where the timestamp allocation is a more complex non-atomic operation that produces behaviors not observed in the atomic case. The original proof by Dodds et al. is a large case analysis that mixes visibility relations with *approximate* LP descriptions, and then adapts and corrects both as the proof advances. However, the axioms and the definitions of the visibility relations have been justified only technically, and have remained unconnected to the intuition behind the structure's design.

By axiomatizing both separability and visibility relations, we derive the following contributions: (1) We obtain a linearizability proof that elides LPs, and is thus more declarative than the proof of Dodds et al.; (2) The proof's declarative nature allows us to first consider the simpler variant of the algorithm with atomic timestamp allocation, and then show that the general variant reduces to the atomic case. The staged proof is more intuitive than if we attempted the general case directly, which is what Dodds et al. do.; (3) Our contribution goes beyond a new proof for the timestamped stack, as it suggests a *systematic* way to axiomatize concurrent data structures in the visibility style. More specifically, we show that the visibility relation naturally emerges when one transforms an obvious state-based sequential axiomatization of stacks to the concurrent setting with histories. In the process, the separability relation also naturally emerges, because one is immediately forced to generalize the returns-before relation. So obtained axioms identify the abstractions that are essential for understanding the algorithm, and strongly guide the remaining proof. Finally, our approach to axiomatization applies to other concurrent algorithms as well, and we comment in Section 5 how we did so for RDCSS and MCAS of Harris et al. [8] and some other structures. Of course, the generality of the approach remains to be evaluated on a wider set of examples.

```
 1: pools : Node[maxThreads];
 2: TS, ID : Int = 0;
 3:
 4: proc push (v : Val)
 5:     Node n =
 6:         Node{v, ∞, pools[TID], false, ID++};
 7:     pools[TID] = n;
 8:     Stamp ts = newTimestamp();
 9:     n.stamp = ts;
10:
11: proc newTimestamp ()
12:     Int ts1 = TS;
13:     pause();
14:     Int ts2 = TS;
15:     if ts1 ≠ ts2 then
16:         return [ts1, ts2 − 1];
17:     else if CAS(TS, ts1, ts1 + 1) then
18:         return [ts1, ts1];
19:     else
20:         return [ts1, TS − 1];
21: record Node:
22:     val : Val; stamp : Stamp; next : Node;
23:     taken : Bool; id : Int ;
24:
25: proc pop ()
26:     Bool suc = false; Node chosen;
27:     while not suc do
28:         Stamp maxT = −∞;
29:         chosen = null;
30:         for i from 0 to maxThreads − 1 do
31:             Node n = pools[i];
32:             while n.taken and n.next ≠ n do
33:                 n = n.next;
34:             Stamp ts = n.stamp;
35:             if maxT <T ts then
36:                 chosen = n; maxT = ts;
37:         if chosen ≠ null then
38:             suc = CAS(chosen.taken, false, true);
39:     return chosen.val;
```

Figure 1 Pseudocode of a simplified TS-stack.

2 The Timestamped Stack and its Timestamps

The timestamped stack (TS-stack) keeps an array of *pools*, indexed by thread IDs; one pool for each thread. A pool is a linked list of nodes. The array index identifying the pool (line 1 in Figure 1) stores the head node of the pool list, and each node (lines 22-23) stores a value *val*, timestamp *stamp*, the *next* node in the list, a boolean *taken* indicating if the value has been taken by some pop, and a unique identifier *id* for the node.[1] Each thread can only insert values in its own pool by allocating a node at the head of the list. A value is logically removed from the pool once its *taken* flag is set to *true*.[2]

The *push* procedure inserts a new node containing the pushed value v into the pool of the executing thread with thread id *TID*. More specifically, in line 6, *push* allocates a new node with the value v, infinite timestamp, *next* pointing to the current head of the pool, taken flag set to *false*, and fresh unique identifier, where *ID*++ denotes an atomic fetch and increment on the global counter *ID*. Then, the new node is set as the new head of the *TID* pool (line 7), a new timestamp is generated (line 8) and assigned to the node (line 9) as a replacement for the original infinity timestamp. We will discuss infinity timestamps and the *newTimestamp* procedure further below.

The *pop* procedure traverses the pools (loop at line 30), searching for an untaken node with a maximal timestamp in the partial order $<_T$, updating the current maximum in the variable *maxT* (lines 35-36). Once a maximal node is found, *pop* attempts to remove it by CAS-ing on its *taken* flag at line 38. The *pop* procedure restarts (loop at line 27) if it was not able to take a maximal node at line 38.

The role of $<_T$ is to endow TS-stack with a LIFO discipline whereby an element with a larger timestamp (i.e, the more recently pushed element), is popped first. In the concurrent setting, however, the meaning of "more recent" is not as straightforward as in the sequential setting, as the definition of linearizability allows that overlapping operations can be linearized

[1] Unique identifiers *id* are ghost code (gray color in Figure 1), introduced solely for use in proofs.

[2] For presentation purposes, we simplified the original algorithm, but treat a more general form in Appendix B.2 [5]. The two versions exhibit the same challenges, and use the sames axiomatization and definitions of visibility and separability. The differences between them are discussed in Section 5.

in either order. In particular, if two invocations of *push* overlapped, they can actually be popped in either order. To reflect this property of linearizability, the order $<_{\mathrm{T}}$ is *partial* as opposed to total. However, to be sequentially sound, it is of essence that if two pushes did *not* overlap, then the more recent push is indeed popped first.[3] This is why the implementation of *newTimestamp* should satisfy the property that two non-overlapping calls to *newTimestamp* produce timestamps that actually *are* ordered by $<_{\mathrm{T}}$.

There are several ways in which one can implement *newTimestamp* to satisfy this property, and Figure 1 shows the particularly efficient variant proposed by Dodds et al. [3]. We will return to this variant promptly. However, for purposes of understanding and proving the algorithm linearizable, one may consider a simpler version whereby timestamps are integers, and *newTimestamp* is implemented to keep a global counter that is *atomically* fetch-and-incremented on each call, returning the current count as the fresh timestamp. Such an atomic implementation results in $<_{\mathrm{T}}$ that is actually a total order, and much simpler to analyze than the efficient variant in Figure 1. We will use the atomic implementation as a stepping stone in our proof; we will prove it linearizable first, and then show that the linearizability argument for the efficient variant *reduces* to the atomic case.

The reason to consider a non-atomic implementation at all is that the atomic one suffers from a performance issue that threads contend on the global timestamp counter. The efficient variant from Figure 1 improves on this by introducing *interval* timestamps of the form $[a, b]$ for integers $a \leq b$, where $[a, b] <_{\mathrm{T}} [c, d]$ holds if $b < c$ in the standard integer order. Obviously, so defined $<_{\mathrm{T}}$ is only a partial order, as it does not order *every* two interval timestamps. Nevertheless, it still suffices for linearizability, because if two push events are assigned overlapping interval timestamps, such events must overlap as well, and thus do not constrain the order in which they are popped in a linearization.

The *newTimestamp* from Figure 1 still keeps a global counter *TS*, as the atomic variant would, but it does not always synchronize accesses to it. In particular, *TS* is first read twice into ts_1 and ts_2 (lines 12 and 14, respectively). In the common case when some thread interfered on *TS* (i.e., $ts_1 \neq ts_2$), the method generates an interval with endpoint $ts_2 - 1$, and terminates without having performed any synchronization. Some synchronization is required only when no interference is detected (i.e., $ts_1 = ts_2$). In that case, *newTimestamp* *CAS*-es over *TS* (line 17), to atomically increment *TS*. As *CAS* is an expensive operation, invoking *pause*() in line 13 increases the probability of interference, and thus decreases the need for *CAS*. If the *CAS* succeeds, an interval with endpoint ts_1 is returned. If the *CAS* fails, some other thread increased *TS*, and the method returns an endpoint $TS - 1$. In all cases, when *newTimestamp* terminates, *TS* has been increased either by the executing thread or by another thread, and the generated interval's endpoint is strictly smaller than the current value of *TS*. Thus, a subsequent non-overlapping invocation of *newTimestamp* will produce an interval that is strictly larger in $<_{\mathrm{T}}$. This ensures that two sequentially non-overlapping pushes generate non-overlapping interval timestamps.

Note that *newTimestamp* could return the *same* interval timestamp for two different overlapping invocations. For example, with initial $TS = 0$, a thread T_1, after reading TS the first time at line 12 (returning 0), waits at line 13 while another thread T_2 fully executes *newTimestamp*, meaning that T_2 increased TS at line 17 and returned timestamp $[0, 0]$. When T_1 resumes, it again reads TS at line 14 (returning 1), and so returns $[0, 0]$.

[3] Further assuming that the pops also did not overlap among themselves or with the pushes.

(A_1) Non-empty *pop* $v :: S \xrightarrow{pop()\ \langle v\rangle} S$ $\quad$ (A_2) Empty *pop* $[\,] \xrightarrow{pop()\ \langle \text{EMPTY}\rangle} [\,]$ $\quad$ (A_3) *push* $S \xrightarrow{push(v)\ \langle tt\rangle} v :: S$

(a) State-based sequential specification.

(B_1) LIFO
$u_1 \lessdot o_1 \wedge u_1 \sqsubset u_2 \sqsubset o_1 \implies \exists o_2.\ u_2 \lessdot o_2 \wedge o_2 \sqsubset o_1$
(B_2) Pop uniqueness
$u \lessdot o_1 \wedge u \lessdot o_2 \implies o_1 = o_2$
(B_3) Dependences occur in the past
$u \lessdot o \implies u \sqsubset o$
$(B_{4.1})$ Non-empty *pop*
$o = pop()\,\langle v\rangle \wedge v \neq \text{EMPTY} \implies \exists u.\ u \lessdot o \wedge v = u.in$
$(B_{4.2})$ Empty *pop*
$o_1 = pop()\,\langle \text{EMPTY}\rangle \implies \forall u.\ u \sqsubset o_1 \implies \exists o_2.\ u \lessdot o_2 \wedge o_2 \sqsubset o_1$
$(B_{4.3})$ *push*
$u = push(\_)\,\langle v\rangle \implies v = tt$

(b) History-based sequential specification. Relation $\lessdot : \text{Ev} \times \text{Ev}$ is abstract, and $u.in$ is event u's input.

Figure 2 State-based and history-based sequential specifications for stacks. Variables u, o, and their indexed variants, range over pushes and pops, respectively.

Finally, $<_T$ is formally augmented with infinite timestamps $-\infty$ and ∞, so that $-\infty <_T t <_T \infty$ for any timestamp t generated by *newTimestamp*. This enables *pop* to start its search with minimum timestamp $-\infty$ (line 28). Similarly, *push* can assign maximum timestamp ∞ to a fresh node (line 6) before assigning it a finite timestamp; an intervening pop could take such a fresh node immediately, as the node is the most recent.

3 Axiomatizing Visibility and Separability

3.1 Sequential History Specifications and Visibility Relations

Following Henzinger et al. [10], we start the development of visibility relations by introducing *history-based specifications* for our data structure. History-based specifications describe relationships between the data structure's procedures in an execution history. They are significantly different from the perhaps more customary state-based specifications that describe the actions of a procedure in terms of input and output state. However, history-based specifications scale better to the concurrent setting, which is why concurrent consistency criteria such as linearizability are invariably defined in terms of execution histories.

In this section we focus on *sequential* histories in order to introduce the idea of *visibility relation* in a simple way, before generalizing to concurrent histories in Section 3.2. A sequential history is a sequence of the form $[proc(in_1)\langle out_1\rangle, \ldots, proc(in_n)\langle out_n\rangle]$, where $proc(in_i)\langle out_i\rangle$ means that $proc(in_i)$ executed *atomically* and produced output out_i. We term *event* each element in a sequential history h, and Ev denotes the set of all events in h.

Figure 2 illustrates the distinction between sequential state-based and history-based specifications for stacks. For the state-based specification in Figure 2a, let us denote by $S \xrightarrow{proc(in)\ \langle out\rangle} S'$ the statement that event *proc* with input *in* executes atomically on stack S, produces output *out* and modifies the stack into S'. Axiom A_1 says that a *pop* removes the top element v from a non-empty stack and returns v. Axiom A_2 says that *pop* returns EMPTY when the stack is empty, leaving the stack unchanged. Axiom A_3 says $push(v)$ inserts v into the stack as the new top element, returning the trivial value tt.

Figure 2b shows the history-based sequential specification for stacks. The specification utilizes the *visibility relation* $\lessdot$ to capture a push-pop causal dependence between events. In particular, $u \lessdot o$ means that "event o pops a value that event u pushed onto the stack". We usually say that u is *visible* to o, or that o *observes* u. Under this interpretation, axioms $B_1, ..., B_{4.3}$ state the following expected properties.[4]

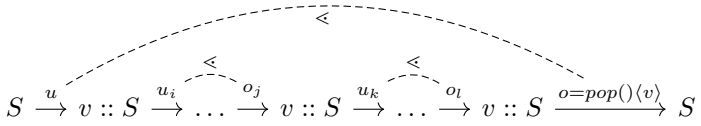
Axiom B_1 (LIFO) states that more recent pushes are popped first. More specifically, if o_1 observes u_1 (i.e, $u_1 \lessdot o_1$) and u_2 is a later push executing between u_1 and o_1 (i.e., $u_1 \sqsubset u_2 \sqsubset o_1$), then u_2 must be popped before o_1 pops u_1, otherwise the value pushed by u_1 would not be at the top of the stack for o_1 to take. Relation $\sqsubset$ is the *returns-before* relation (with $\sqsubseteq$ its reflexive closure), where $x \sqsubset y$ means that x terminated before y started. Note that $\sqsubset$ is a total order on events, as in a sequential execution, different events cannot overlap.

Axiom B_2 (Pop uniqueness) says that a push is observed by at most one pop.

Axiom B_3 (Dependences occur in the past) says that if a pop depends on a push, then the push executes before the pop.

Axioms $B_{4.1}$ (Non-empty *pop*), $B_{4.2}$ (Empty *pop*), and $B_{4.3}$ essentially are the counterparts of the state-based sequential axioms A_1-A_3, respectively, as we show next.

Axiom $B_{4.1}$ says that a $pop()\langle v\rangle$ event o observes a push u that pushed v. This axiom, along with B_1-B_3, ensures that o relates to u as in the following diagram.



In particular: (i) u executes before o (by axiom B_3, because $u \lessdot o$), (ii) every push between u and o is popped before o (by axiom B_1), and each push is popped exactly once (by axiom B_2). Thus, once o executes, the value v pushed by the observed u, is actually the most recent unpopped value, i.e., it is on the top of the stack. Subsequent pops cannot observe this value anymore either (again by axioms B_1 and B_2), thus the stack is modified from $v :: S$ to S. This explains that $B_{4.1}$ is essentially a history-based version of A_1.

Similarly, axiom $B_{4.2}$ states that if a $pop()\langle \text{EMPTY}\rangle$ event o occurs, then every push before o must have been popped before o, as this ensures that the stack is empty when o is reached. Hence, the axiom is counterpart to A_2.

Finally, axiom $B_{4.3}$ says that the output of a push event is the trivial value tt. The axiom imposes no conditions on the stack, as a value can always be pushed. In this, $B_{4.3}$ is the counterpart to A_3 which also imposes no conditions on the input stack, and posits that push's output value is trivial. However, unlike A_3, $B_{4.3}$ does not directly says that the value is pushed on the top of the stack, as that aspect is captured by the relationships between pushes and pops described by $B_{4.1}$.

3.2 Concurrent Specifications and Separability Relations

Concurrent execution histories do not satisfy the sequential axioms in Figure 2b for two related reasons. First, concurrent events can *overlap* in real time. As a consequence, the axioms B_1 (LIFO), B_3 (Dependencies occur in the past), and $B_{4.2}$ (Empty *pop*) are too

[4] Our paper will make heavy use of several different relations. To help the reader keep track of them, we denote the relations by symbols that graphically associate to the relation's meaning. For example, we use $\lessdot$ for the visibility relation, because the symbol graphically resembles an eye.

(C_1) Concurrent LIFO
$u_1 \lessdot o_1 \wedge o_1 \not\preceq u_2 \not\preceq u_1 \implies \exists o_2.\ u_2 \lessdot o_2 \wedge o_2 \ltimes o_1$

(C_2) Pop uniqueness
$u \lessdot o_1 \wedge u \lessdot o_2 \implies o_1 = o_2$

(C_3) No future dependences
$x \prec^+ y \implies y \not\sqsubseteq x$

(C_4) Return value completion
$\exists v.\ \mathcal{Q}_{x,v} \wedge (x \in T \implies v = x.out)$

(a) Concurrent specification. Relations $\lessdot, \ltimes : \text{Ev} \times \text{Ev}$ are abstract.

Constraint relation
$\prec \ \widehat{=}\ \lessdot \cup \ltimes$

Returns-before relation
$e_1 \sqsubset e_2 \ \widehat{=}\ e_1.end <_{\mathbb{N}} e_2.start$

Set of terminated events
$T \ \widehat{=}\ \{e \mid e.end \neq \bot\}$

Closure of terminated events
$\overline{T} \ \widehat{=}\ \{e \mid \exists t \in T.\ e \prec^* t\}$

$$\mathcal{Q}_{o_1,v} \ \widehat{=}\ \begin{cases} \exists u.\ u \lessdot o_1 \wedge v = u.in & \text{if } v \neq \text{EMPTY} \\ \forall u.\ o_1 \not\preceq u \implies \exists o_2.\ u \lessdot o_2 \wedge o_2 \ltimes o_1 & \text{if } v = \text{EMPTY} \end{cases} \qquad \mathcal{Q}_{u,v} \ \widehat{=}\ v = tt$$

(b) Defined notions.

Figure 3 Concurrent history-based specification for stacks. Variables u, o, and their indexed variations, range over pushes and pops in $\overline{T}$, respectively. Variables x, y range over $\overline{T}$. Variable e, and its indexed variations, range over Ev. $x.out$ denotes x's output.

restrictive, as they force events to be non-overlapping (i.e., disjoint in time) due to the use of the returns-before relation $\sqsubset$. Second, events can no longer be treated as atomic; thus event's start and end times (if the event terminated) must be taken into account. As a consequence, axioms $B_{4.1}$, $B_{4.2}$, and $B_{4.3}$ must be modified to account for the output of an unfinished event not being available yet. We continue using Ev for the set of events in the concurrent history. We denote by $e.start$ and $e.end$ the start and end time of event e, respectively; for example, for the implementation in Figure 1, a *push* event starts when line 5 executes, and ends when line 9 executes. We use the standard order relation on natural numbers $<_{\mathbb{N}}$ to compare start and end times.

Figure 3 shows the modified axioms that address the above issues. Importantly, in addition to the visibility relation, the axioms utilize the *separability relation* $x \ltimes y$ to capture that "event x is separable before y", i.e., x should be linearized before y.[5] The reason for the separation depends on the particular stack implementation, but is kept abstract in the axioms. Correspondingly, the relation $\ltimes$ is also kept abstract. We now explain how the concurrent axioms in Figure 3 are *systematically* obtained from the sequential ones in Figure 2b.

Axiom C_1 is obtained from B_1 by replacing $\sqsubset$ with $\ltimes$ or with the (negation of the) reflexive closure $\not\preceq$, following the rules below. The goal is to relax the real-time strong separation imposed by $\sqsubset$ with a more permissive separation of $\ltimes$.

- If subformula $a \sqsubset b$ occurs in a condition of an implication (negative occurrence), it is replaced with $b \not\preceq a$. Notice the flip in the arguments and the negation.
- If subformula $a \sqsubset b$ occurs in the conclusion of an implication (positive occurrence), it is replaced with $a \ltimes b$.

These rules have the following justification. Let us suppose we have a formula $\phi \ \widehat{=}\ a \sqsubset b \implies c \sqsubset d$ in some sequential axiom. In the sequential case, $\sqsubset$ is a total order, which means that ϕ is equivalent to $b \sqsubseteq a \vee c \sqsubset d$. After directly replacing $\sqsubset$ for $\ltimes$, we obtain

[5] The symbol $\ltimes$ twists $\sqsubset$, suggesting that $\ltimes$ relaxes (i.e., is a twist on) returns-before relation $\sqsubset$.

$b \underline{\ltimes}\ a \vee c \ltimes d$, which is further equivalent to $\psi \ \widehat{=}\ b \not\underline{\ltimes}\ a \implies c \ltimes d$. Comparing ϕ and ψ, we see that ψ's condition is flipped, replaced, and negated, while its conclusion is only replaced. An important aspect of our procedure is that negative occurrences of $\ltimes$ in ψ are themselves negated. Thus, intuitively, ψ as a whole remains positive with respect to $\ltimes$. Positive formulas remain true under extensions of $\ltimes$, which is crucial, as the linearizability proof will involve extending $\ltimes$ until reaching a total order.

Axiom C_2 is unchanged compared to B_2.

Axiom C_3 is obtained from B_3 as follows. In the sequential specification, $\lessdot$ was the only relation encoding dependences between events, but now we have two relations encoding dependences, $\lessdot$ and $\ltimes$. To collect them, we define a new relation $\prec \ \widehat{=}\ \lessdot \cup \ltimes$ which we call *constraint* relation.[6] We can consider modifying Axiom B_3 into $x \prec y \implies x \sqsubset y$ to say that any dependence x of y must terminate before y starts. However, such a modification of B_3 is too stringent, as it does not allow x to overlap with y. Instead, we relax the conclusion to say that an event cannot depend on itself or events from the future, i.e., $x \prec y \implies y \not\sqsubseteq x$. Finally, we get axiom C_3 by replacing $\prec$ with its transitive closure $\prec^+$ to account for *indirect* dependences of y; e.g., in $x_1 \prec x_2 \prec y$, event x_1 is an indirect dependence of y. Hence, C_3 reads "any direct or indirect dependence does not execute in the future, and events do not depend on themselves".

To understand Axiom C_4, we need to consider the set T of all *terminated* events and its closure under the constraint relation $\overline{T} \ \widehat{=}\ \{e \in \textsc{Ev} \mid \exists t \in T.\ e \prec^* t\}$. As usual, $\prec^*$ is the reflexive-transitive closure of $\prec$. The reason for considering this set is that the variable x over which the axiom implicitly quantifies ranges over $\overline{T}$.

It is standard in linearizability that the linearization order contains all the terminated events, plus selected unterminated events with fictitious, but suitable, outputs. The selected unterminated events are typically those that executed their effect, which then influenced others, and must thus be included for sequential soundness. The set $\overline{T}$ precisely determines the events to be included by saturating the set of terminated events T under $\prec$.

Axiom C_4 then codifies when an output v is suitable for an event x by means of the *postcondition predicate* $\mathcal{Q}_{x,v}$. In particular, C_4 says that v exists such that $\mathcal{Q}_{x,v}$. If $x \in \overline{T}$ is unterminated, we use that v as the fictitious output. If x is terminated ($x \in T$), then v must be x's actual output. The postcondition predicate $\mathcal{Q}_{x,v}$ describes how x and v relate in the case of stacks. It is obtained by coalescing the axioms $B_{4.1}$, $B_{4.2}$ and $B_{4.3}$, which themselves describe the outputs of stack events in the sequential setting, and which we first modify according to the systematic transformation outlined above.

We henceforth call the axioms in Figure 3, *visibility-style axioms*. These axioms imply linearizability of any stack implementation satisfying them,

▶ **Theorem 3.1.** *Let D be an arbitrary implementation of a concurrent stack. If there are relations $\ltimes$ and $\lessdot$ definable using D such that the visibility-style axioms hold, then D is linearizable.*

The proof starts with the relation $\lhd \ \widehat{=}\ (\prec \cup \sqsubset)^+$, i.e, the transitive closure of the union of $\prec$ and $\sqsubset$. Then, it shows that $\lhd$ is a partial order that can be extended to a sequentially sound total order $\leq$ by using the visibility-style axioms. Since $\leq$ contains $\prec$, this means that relations $\lessdot$ and $\ltimes$ define ordering constraints that linearization respects.

[6] The symbol $\prec$ is like an eye with no iris; thus, "blinder" than $\lessdot$, reflecting that $\prec$ is a superset of $\lessdot$.

4 Visibility and Separability for the TS-stack

By Theorem 3.1, to prove linearizability for the TS-stack, it suffices to define the relations $\lessdot$ and $\ltimes$ and show that they satisfy the axioms from Figure 3. We carry out this proof in two stages: In Section 4.2 we prove linearizability when the *newTimestamp* procedure is implemented by an atomic fetch-and-increment operation on a global counter *TS*, as discussed in Section 2. In Section 4.3 we show how the general case of interval timestamps reduces to the atomic case. The lifting exploits that the difference between the atomic and interval cases is only in the implementation of *newTimestamp*.

For simplicity, in both cases we explicitly exclude *elimination pairs* from the discussion. An elimination pair consists of a push and an overlapping pop event that takes the value pushed. The elision allows the discussion to only consider pushes with *finite* timestamps. Indeed, every push is first assigned an infinite timestamp (line 6 in Figure 1), which is then refined into a finite one in line 9. If a push u, having not yet reached line 9, is taken by some pop o, then u and o overlap, and hence form an elimination pair.[7]

Also, in both cases, we utilize the abstraction we call *spans*, to define the visibility and separability relations. A span of an event is the interval in which the event accesses the shared state of the stack. We could trivially take the span to be the whole interval of the event, but in the case of TS-stack we can tighten it as discussed below. In this sense, a span is a generalization of LPs; being an interval, rather than a single point, it *approximates* where the LP of an event lies, but allows for some uncertainty as to the LPs exact position.

The span of the *push* procedure starts when the new node is linked as the first node of the pool (line 7), as this is the moment when the new node becomes available for other events to see. The span ends when a finite timestamp is assigned to the new node (line 9). Notice how the span encompasses all the commands of *push* that change the pool or the new node.

The span of the *pop* procedure starts at the infinity stamp assignment (line 28) of the last iteration of the pools scan. The span ends at the successful CAS at line 38 which takes the node for the pop to return. Again, the span covers all the commands of *pop* that change the pools or the taken node. These are all included in the last iteration of the pools scan, because in all the prior iterations, the CAS modifying the pools must have failed.

We formalize spans as pairs of rep events (a, b), where a and b are the initial and final rep event in the span, respectively. We denote by $\mathit{start}\,(b)$ and $\mathit{end}\,(b)$ the standard projection functions for span b. Rep events are generated by the invocation of a code line inside a procedure. For example, invoking line 7 in Figure 1 produces a rep event. The set of all rep events in an execution history is denoted as Rep. The distinction between events (Ev) and rep events is standard in linearizability [11]. We denote by $<$ the real-time order between rep events. We consider only fully-formed spans; for example, if *pop* has not executed its successful CAS, then it has no span.

We also extend our notion of timestamp into *abstract timestamp*. An abstract timestamp is a pair (i, t), where i is the (ghost) id of a node, and t is a (plain) timestamp. The extension is motivated by the observation explained in Section 2 that two pushes may actually generate

[7] Eliding elimination pairs when dealing with stacks is justified because such pairs can be linearized simply as a push that is immediately followed by a pop. The idea was originated by Hendler et al. [9] and was also employed in Haas' PhD dissertation [7], though with a different motivation from us and with a different soundness proof. For example, to prove the elimination sound, Haas shows how elimination pairs could be put back into the histories from which they have been removed. In contrast, we define visibility and separability relations that exclude elimination pairs, and show in Appendix B.1 [5], how to extend the relations iteratively, one elimination pair at a time. The extension adds some bulk, but does not change the structure of the proof that we illustrate in this section.

the same (plain) timestamp. By attaching the node id i to the timestamp t, we differentiate such cases. We use "timestamp" to refer to abstract timestamps or plain timestamps when the adjective can be inferred from the context.

We also utilize the following notation.

- Given event e, $\mathcal{S}\ e$ is the unique span executed by e. The function is undefined if the argument event has not completed its span.
- Given spans a, b, the relation $a \sqsubset^S b$ means that a finished before b started. $\sqsubseteq^S$ denotes its reflexive closure.
- For a push u, $id\ u$ is the unique id of the node that u inserted into the pool in line 7. Similarly, for a pop o, $id\ o$ is the unique id of the node that o took at the successful CAS in line 38. If u and o have not executed the mentioned lines, id is undefined.
- For a push u, $t_s\ u$ is the abstract timestamp $(id\ u, t)$, combining $id\ u$ with the timestamp t that u assigned at line 9. In particular, t is always *finite*, because *newTimestamp* only generates finite timestamps. Similarly, for a pop o, $t_s\ o$ is the abstract timestamp $(id\ o, t)$, combining $id\ o$ with the timestamp t of the taken node that o read in line 34. Generally, $t_s\ o$ may return an infinite plain timestamp; however, if elimination pairs are excluded, then timestamps are finite, as explained before. If an event x has not executed its span, $t_s\ x$ is undefined.
- Abstract timestamps admit the following partial order defined out of $<_{\mathrm{T}}$ on plain timestamps, where we overload the symbol $<_{\mathrm{T}}$ without confusion.

$$(i_1, t_1) <_{\mathrm{T}} (i_2, t_2) \ \widehat{=}\ t_1 <_{\mathrm{T}} t_2$$

- We define when push u and pop o form an elimination pair.

$$u \ \textsc{Elim}\ o \ \widehat{=}\ id\ u = id\ o \wedge u \not\sqsubset o$$

  In English: (1) o pops the node that u pushed ($id\ u = id\ o$), and (2) u and o overlap. Events u and o overlap if $u \not\sqsubset o$ and $o \not\sqsubset u$, but it is not necessary to explicitly check $o \not\sqsubset u$, as that follows from $id\ u = id\ o$ and a structural invariant that o cannot pop a node that has not been pushed yet (Appendix B.1 [5]).
- The set of events that occur in elimination pairs is $E \ \widehat{=}\ \{x \mid \exists y.\ x \ \textsc{Elim}\ y \vee y \ \textsc{Elim}\ x\}$. As we explicitly exclude elimination pairs from the presentation, we assume that each event variable x occurring in the forthcoming definitions is such that $x \notin E$. In Section 5 we comment how elimination pairs are placed back into consideration.

4.1 Key Abstractions and Invariants

When pop misses a push. The key for understanding TS-stacks is explaining what it means for a pop o to have *missed* a push u. Informally, a miss occurs when o, in its scan of the pools, takes a push u' with a smaller timestamp than that of u (hence, u' is less recent than u). This is critical, because o taking a less recent push than available is seemingly a violation of the LIFO order. However, this does not actually have to be so in the case of TS-stacks, where, for example, it is fine for u to insert into the pool after o has already scanned past the point of insertion. We can say that u occurred too late to really be available for o to pop, and we simply linearize u after o. The following definition formalizes when o misses u (i.e., when u occurs too late for o), focusing on the atomic timestamp case.

$$\begin{aligned} \textsc{Miss}\ o\ u \ \widehat{=}\ & t_s\ o\ <_{\mathrm{T}}\ t_s\ u \wedge \\ & \forall o'.\ t_s\ u = t_s\ o' \implies end\,(\mathcal{S}\ o) < end\,(\mathcal{S}\ o') \end{aligned} \tag{1}$$

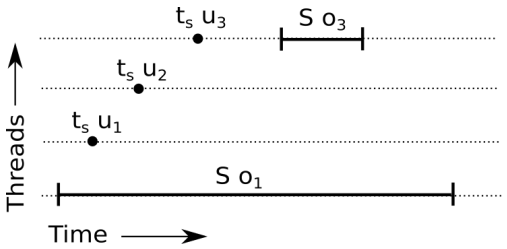


Figure 4 Possible execution showing three atomic timestamp generation rep events for push events u_1, u_2, u_3, labeled by their generated timestamps; and two pop spans for pop events o_1 and o_3. Spans are shown as line segments and rep events (being atomic) as dots. The timestamps are strictly increasing $t_s\ u_1 <_{\mathrm{T}} t_s\ u_2 <_{\mathrm{T}} t_s\ u_3$. Event o_1 took u_1, while o_3 took u_3. The events will be linearized as u_1, o_1, u_2, u_3, o_3. In particular, o_1 must be linearized before u_2.

The first conjunct directly says that for o to miss u, it must be that o takes a node with a smaller timestamp than that of u. The second conjunct adds that, intuitively, u *remains untaken* during the execution of o. Indeed, if u is taken by o' ($t_s\ u = t_s\ o'$), the definition requires that the span of o' finishes after the span of o ($\mathit{end}\,(\mathcal{S}\ o) < \mathit{end}\,(\mathcal{S}\ o')$). That is, the CAS that sets the *taken* flag in the node of u executes after the span of o. In other words, if u is taken at all, then it is taken *after* the span of o.

Figure 4 shows a push u_2 that overlaps with a pop o_1, but o_1 takes a push u_1 whose timestamp is smaller than that of u_2. In our definition, Miss $o_1\ u_2$ holds because u_2 remains untaken on the stack after o_1 terminates. Miss $o_1\ u_2$ indicates that we must linearize o_1 before u_2. And indeed, this is consistent with the situation in the figure, as any order where u_2 appears before o_1 violates some linearizability requirement. For example, the order u_1, u_2, o_1 is sequentially unsound because o_1 pops u_1 while u_2 is the top of the stack, while u_2, u_1, o_1 does not respect the ordering of the timestamps of u_1 and u_2.[8]

Continuing with Figure 4, o_1 *does not miss* u_3, even though u_3 also overlaps with o_1, and o_1 takes u_1 whose timestamp is smaller than that of u_3. In our definition, ¬Miss $o_1\ u_3$ because the span of o_3 ends before the span of o_1. ¬Miss $o_1\ u_3$ indicates no restrictions on the ordering between o_1 and u_3. For example, the only linearization order of Figure 4 is u_1, o_1, u_2, u_3, o_3, but this is forced by the existence of u_2. Removing u_2, the orders u_1, u_3, o_3, o_1 (where u_3 appears before o_1) and u_1, o_1, u_3, o_3 (where u_3 appears after o_1) are both valid.

Misses start late. Having defined Miss $o\ u$, we can now explain the most important invariants of the TS-stack, again focused on the atomic timestamps. The first invariant says that a push u missed by a pop o has a span that starts *after* the pop's span starts. In other words, a missed push starts after the pop that missed it.

$$\text{Miss}\ o\ u \implies \mathit{start}\,(\mathcal{S}\ o) < \mathit{start}\,(\mathcal{S}\ u) \tag{2}$$

To intuit why (2) is an invariant, consider a situation when o misses u but u's span starts before o's. In that case, u's pool contains u's node *before* o even starts its scan. Thus o's scan will encounter u and proceed to either take u, or take an even more recent push. At any rate, o will not take a push with a timestamp below that of u; thus, ¬(Miss $o\ u$).

[8] We linearize pushes by the order of their timestamps.

Disjoint pushes order timestamps. The next invariant is that pushes with disjoint spans, produce ordered timestamps. Intuitively, this is so because disjoint push spans make disjoint calls to *newTimestamp*, which in turn generate ordered timestamps as explained in Section 2.

$$\mathcal{S}\ u_1 \sqsubset^S \mathcal{S}\ u_2 \implies t_s\ u_1 <_{\mathrm{T}} t_s\ u_2 \tag{3}$$

4.2 Case: Atomic Timestamps

We next define the visibility $\lessdot$ and separability $\ltimes$ relations for atomic timestamps.

$$u \lessdot o \mathrel{\widehat{=}} t_s\ u = t_s\ o \tag{4}$$

$$u_1 \ltimes u_2 \mathrel{\widehat{=}} t_s\ u_1\ <_{\mathrm{T}}\ t_s\ u_2 \tag{5}$$

$$o \ltimes u \mathrel{\widehat{=}} \exists u'.\ \textsc{Miss}\ o\ u'\ \wedge\ t_s\ u'\ \leq_{\mathrm{T}}\ t_s\ u \tag{6}$$

$$o_2 \ltimes o_1 \mathrel{\widehat{=}} t_s\ o_1\ <_{\mathrm{T}}\ t_s\ o_2\ \wedge\ \neg\exists u'.\ \textsc{Miss}\ o_1\ u'\ \wedge\ t_s\ u'\ \leq_{\mathrm{T}}\ t_s\ o_2 \tag{7}$$

The definition of $\lessdot$ relates u and o if they have the same timestamp (i.e., o took u).

The definition of $\ltimes$ comes with three clauses, motivated by the form of the axioms from Figure 3. In particular, we need to separate a push from a push ($u_1 \ltimes u_2$), a pop from a push ($o \ltimes u$), and a pop from a pop ($o_2 \ltimes o_1$), but not a push from a pop, as only the first three clauses of $\ltimes$ appear in the axioms.

The clause $u_1 \ltimes u_2$ naturally orders push events according to their timestamps.

The clause $o \ltimes u$ extends $\textsc{Miss}\ o\ u'$ to account for pushes being ordered by their timestamps, as per the previous clause. It says that pop o is separated before push u, if there is a push u' that was missed by o, and the timestamp of u' is below (or equals) that of u. For example, in Figure 4 we have $o_1 \ltimes u_2$ and $o_1 \ltimes u_3$.

The clause $o_2 \ltimes o_1$ separates pops inversely to the order of the taken timestamps, or equivalently, inversely to the order of the taken pushes, but under the condition that o_1 *did not miss any push with a timestamp below* o_2. The last requirement is important. For example, if we ignored it in Figure 4, we would obtain $o_3 \ltimes o_1$ since $t_s\ u_1 <_{\mathrm{T}} t_s\ u_3$. But this order is sequentially unsound; the pushes being ordered as u_1, u_2, u_3, after o_3 takes u_3, the value pushed by u_2 is at the top of the stack. But then o_1 cannot execute next, as we need an intervening pop to remove u_2.

It is worth mentioning that we arrived at the definition of the clause $o_2 \ltimes o_1$ by formal symbol manipulation aimed at fulfilling axiom C_1 (Concurrent LIFO) after the definitions of the other clauses have been unfolded in C_1. In hindsight, this may have been expected, as the clauses $u_1 \ltimes u_2$ and $o \ltimes u$ are hypotheses of C_1, while $o_2 \ltimes o_1$ is in the conclusion.

The engineering of the (uniquely determined) definition of the clause $o_2 \ltimes o_1$ thus makes the proof of axiom C_1 out of definitions (4)-(7) quite straightforward, but for one important observation. Because the axiom contains negations of several clauses of $\ltimes$, unfolding the definitions of these clauses reveals comparisons of the form $t_s\ x \not<_{\mathrm{T}} t_s\ y$, where the relation $<_{\mathrm{T}}$ appears negated. The proof then crucially relies on $<_{\mathrm{T}}$ being total, so that we can *flip* the negated comparisons into the form $t_s\ y\ \leq_{\mathrm{T}}\ t_s\ x$. It is the requirement of totality of $<_{\mathrm{T}}$ that makes the described development specific to atomic timestamps. In Section 4.3, we shall see how to adapt to interval timestamps where $<_{\mathrm{T}}$ is *not* total.

▶ **Theorem 4.1.** *Given $\lessdot$ and $\ltimes$ as in (4)–(7), the TS-stack with atomic timestamps satisfies the invariants in Section 4.1 and the axioms in Figure 3, and is thus linearizable by Theorem 3.1.*

The characteristic part of the proof is showing that the axiom C_3 (no future dependences) holds, which is where we rely on the invariants (2) and (3). This proof generates obligations, one of which is that $o \ltimes u \sqsubset o$ for some push u and pop o is impossible, as such u depends on o which is in u's future. The proof proceeds by contradiction: suppose $o \ltimes u \sqsubset o$. By definition of $o \ltimes u$, there exists a push u' missed by o such that $t_s\ u' \leq_{\mathrm{T}} t_s\ u$. By invariant (2), u' starts after o starts, and since $u \sqsubset o$, it must also be $u \sqsubset u'$. But then, by invariant (3), it is also $t_s\ u <_{\mathrm{T}} t_s\ u'$. In other words, $t_s\ u <_{\mathrm{T}} t_s\ u' \leq_{\mathrm{T}} t_s\ u$, a contradiction.

4.3 Case: Interval Timestamps

The proof from Section 4.2 does not directly apply to the interval timestamps because proving axiom C_1 (Concurrent LIFO) relies on the totality of $<_{\mathrm{T}}$ in order to flip the negated inequalities $t_s\ x \not<_{\mathrm{T}} t_s\ y$ into positive facts $t_s\ y\ \leq_{\mathrm{T}}\ t_s\ x$. The relation $<_{\mathrm{T}}$ is total in the atomic case, but not in the interval case.

The key observation that allows us to recover the argument is that *whenever $<_{\mathrm{T}}$ is used to compare the timestamps of two push events in the proofs of the atomic case, at least one of the push events is invariably popped.* In other words, the proof does not actually require totality, but only the following weaker property of *pop-totality*. Formally, if R is a partial order on abstract timestamps, then R is pop-total if:

$$\begin{aligned}\forall u_1\ u_2\ o.\, (t_s\ u_1 = t_s\ o) \vee (t_s\ u_2 = t_s\ o)\ \Longrightarrow\\ (t_s\ u_1)\ R\ (t_s\ u_2) \vee (t_s\ u_2)\ R\ (t_s\ u_1) \vee (t_s\ u_1 = t_s\ u_2)\end{aligned} \tag{8}$$

In English: if at least one of the pushes is taken, then the timestamps generated by the pushes are totally comparable under R.

As an illustration why the weaker property suffices, consider the hypotheses of the axiom C_1: these are $u_1 \lessdot o_1$, $o_1 \not\trianglelefteq u_2$ and $u_2 \not\trianglelefteq u_1$. Let us further assume that $<_{\mathrm{T}}$ in all the definitions is replaced by an arbitrary pop-total R. A common pattern throughout the proof of Theorem 4.1 is that three conjuncts of the above form appear together. Such combination entails that u_1 and u_2 are both popped, thus allowing us to flip any negated relation R in which $t_s\ u_1$ or $t_s\ u_2$ may appear.

Indeed, that u_1 is popped, and by o_1, follows from $u_1 \lessdot o_1$, which is defined as $t_s\ o_1 = t_s\ u_1$. To see that u_2 must also be popped consider the following. First, note that $o_1 \not\trianglelefteq u_2$ implies $\neg\textsc{Miss}\ o_1\ u_2$, by an easy derivation. Pushing the negation inside the definition of Miss and substituting $t_s\ o_1 = t_s\ u_1$ derives $\neg(t_s\ u_1)\ R\ (t_s\ u_2) \vee \exists o'.\, t_s\ u_2 = t_s\ o' \wedge \ldots$. The second disjunct directly says that u_2 is popped by some o'. By pop-totality of R, the first disjunct implies $(t_s\ u_2)\ R\ (t_s\ u_1) \vee (t_s\ u_2) = (t_s\ u_1)$, and thus $t_s\ u_2 = t_s\ u_1$, because $\neg(t_s\ u_2)\ R\ (t_s\ u_1)$ by $u_2 \not\trianglelefteq u_1$. Thus, u_1 and u_2 are the same push, and u_2 is popped as well.

It follows that we could replicate the atomic case proof to the interval case, if we could replace $<_{\mathrm{T}}$ with some pop-total relation over *interval timestamps* throughout the definitions and proofs in Sections 4.1 and 4.2. We next define such a relation $\ll$ that includes $<_{\mathrm{T}}$.

$$\begin{aligned}t_2 \ll t_1\ &\widehat{=}\ t_2 <_{\mathrm{T}} t_1\ \vee\ \exists u_1, u_2.\ t_s\ u_1 \not<_{\mathrm{T}} t_s\ u_2\ \wedge\ \textsc{Tb}\ u_1\ u_2\ \wedge\\ &\qquad\qquad t_2 \leq_{\mathrm{T}} t_s\ u_2\ \wedge\ t_s\ u_1 \leq_{\mathrm{T}} t_1\\ \textsc{Tb}\ u_1\ u_2\ &\widehat{=}\ \exists o_1.\ t_s\ u_1 = t_s\ o_1\ \wedge\ \forall o_2.\ t_s\ u_2 = t_s\ o_2\ \Longrightarrow\ end\,(\mathcal{S}\ o_1) < end\,(\mathcal{S}\ o_2)\end{aligned}$$

The key insight of the definition is that if two pushes u_1 and u_2 are not already ordered by $<_{\mathrm{T}}$, i.e., $t_s\ u_1 \not<_{\mathrm{T}} t_s\ u_2$, we could order their timestamps in $\ll$ in the order in which the pushes are popped. Indeed, if u_1 is *taken before* u_2 ($\textsc{Tb}\ u_1\ u_2$), then LIFO warrants that u_2

is linearized before u_1. We thus order u_2's timestamp before u_1's timestamp in $\ll$. It follows that $u_2 \ltimes u_1$ (assuming $\ll$ substitutes $<_T$ in the definition of $u_2 \ltimes u_1$), and consequently that u_2 is linearized before u_1. The definition of $\ll$ further saturates the relation to include any $t_2 \ll t_1$ where $t_2 \leq_T t_s\ u_2$ and $t_s\ u_1 \leq_T t_1$, as then $t_2 \ll t_1$ is forced by $t_s\ u_2 \ll t_s\ u_1$.

Returning to taken-before, we define $\textsc{Tb}\ u_1\ u_2$ to hold of two pops u_1 and u_2 if: (1) u_1 is taken and u_2 is not, or (2) both are taken by pops o_1 and o_2, respectively. In the case (2) we require that the span of o_1 ends before the span of o_2, i.e., o_1 took its push before o_2 did.

One can now proceed to prove that $\ll$ is a strict partial order that is pop-total, that the invariants "Misses start late" and "Disjoint pushes order timestamps" from Section 4.1, continue to hold for the TS-stack with interval timestamps, after substituting $<_T := \ll$ in the definition of $\textsc{Miss}$ (1), and definitions (2), (3) of the invariants. The visibility and separability relations for the TS-stack with interval timestamps are exactly as in (4)-(7) but with substitution $<_T := \ll$, and our final theorem about the correctness of TS-stack is obtained simply by retracing the proof of Theorem 4.1.

▶ **Theorem 4.2.** *Let $\lessdot$ and $\ltimes$ defined as in (4)–(7) but under the substitution $<_T := \ll$. The TS-stack with interval timestamps satisfies the invariants in Section 4.1 under the substitution, and the axioms in Figure 3, and is thus linearizable by Theorem 3.1.*

5 Discussion, Related and Future Work

Dealing with elimination pairs. To handle elimination pairs that were excluded in Section 4, we recursively define indexed families of visibility and separability relations, where $\lessdot_i$ and $\ltimes_i$ means that the first i elimination pairs have been added (Appendix B.1 [5]). At level 0, $\lessdot_0$ and $\ltimes_0$ are the relations from Section 4. At some limit level n, where n is the number of elimination pairs, we have the final relations $\lessdot_n$ and $\ltimes_n$ that consider all the events.

The theorems in Section 4 show that the visibility-style axioms in Figure 3 hold for events in $\overline{T} \setminus E$, i.e., $\overline{T}$ without elimination pairs. They are the base case of our proof in Appendix B.1 [5], which proceeds to inductively show that if the visibility-style axioms hold for the first i pairs, they continue to hold when the pair $i+1$ is added.

Differences with the original algorithm. Figure 1 is a simplified version of the algorithm from Appendix B.2 [5]. The latter further treats elimination pair detection and node unlinking (i.e., node deallocation from memory). We consider the simplified version solely for presentation reasons, as the simplification still presents the same verification challenges and suffices to motivate the visibility and separability relations in Section 4. The definitions of these relations transfer to Appendix B.1 [5], where they serve as a basis for defining a family of augmented relations that deal with elimination pairs, as described above.

Having said this, the program that we treat in Appendix B.2 [5] still differs in a relatively minor way from the original program of Dodds et al. [3] in that we elide empty stack detection (i.e. pops returning EMPTY). This can be treated separately as an extra independent step in the proof [7], which means that considering empty pops changes neither the analysis we already presented in Section 4 nor the proof for elimination pairs in Appendix B.1.3 [5]. Nevertheless, we plan to augment the proof with an extra step that considers empty pops.

Related proofs. Dodds et al. [3] proof is also based on a visibility relation (their **val**), in addition to several other relations. However, our two axiomatizations and proofs differ significantly. Our axiomatization arises from a systematic transformation of a state-based sequential specification of stacks into a history-based concurrent specification, while that

of Dodds et al. does not seem to derive from such prior principles, though it does suffice for the linearizability proof. The different axiomatizations give rise to different relations on histories as well. For example, their insert-remove (**ir**) relation is defined in terms of LPs of submodules. The objective in using LPs of submodules is to start with a definition for that may have linearizability violations, which then gets adjusted along the proof to remove such violations. In contrast, the definitions of our relations in Section 4 require no adjustments since they already lead to a correct linearization, albeit by eliding LPs. As a result, our relations are quite a bit more direct, and support better proof decomposition. In particular, our proof transfers from the easier atomic timestamp case to the more difficult interval timestamp case, whereas Dodds et al. immediately consider the interval case.

Bouajjani et al. [1] employs forward simulation on the atomic timestamp variant of the TS-stack, but do not attempt the interval timestamp variant. Our proof (Appendix B.2.2 [5]) does not employ simulations, and also lifts the atomic timestamp case to the interval timestamp case. The lifting exploits that the difference between the atomic and interval timestamp cases is not in the program structure, but only in the implementation of *newTimestamp*.

Visibility relations in other contexts. Our approach uses visibility and separability relations to model ordering dependencies between events. A general survey of the use of visibility relations in concurrency and distributed systems is given by Viotti and Vukolić [17]. Visibility relations and declarative proofs have also been utilized to specify consistency criteria weaker than linearizability (Emmi and Enea [6]), to introduce a specification framework for weak memory models (Raad et al. [15]), and to specify the RC11 memory model (Lahav et al. [13]).

In contrast to the above papers that focus on the semantics of consistency criteria, our use of visibility relations focuses on verifying specific algorithms and data structures, and is thus closer to the following work where visibility relations are applied to concurrent queues (Henzinger et al. [10, 2]), concurrent stacks (Dodds et al. [3] and Haas [7]), and memory snapshot algorithms (Öhman and Nanevski [14]). We differ from these in the addressed structures, or in the case of Dodds et al. in the structure of the proof and its components.

Our key innovation compared to these works is the introduction of the separability relation and its utilization to systematically axiomatize the stack structure in a novel way.

Visibility and separability as a general methodology. The pattern suggested by Sections 3.1 and 3.2, whereby one transforms a history-based sequential specification into a concurrent specification, by replacing the returns-before relation $\sqsubset$ with a separability relation $\ltimes$, points towards a general methodology for axiomatizing concurrent structures.

To test the generality of the approach, we have applied it – successfully ([4] and Appendix C.1 [5]) – to the RDCSS and MCAS algorithms of Harris et al. [8]. These algorithms write *descriptors* (a record with information about the task that a thread requires help with) into pointers, so that a thread that reads a descriptor can provide help by executing the described task. These algorithms implicitly "bunch" their help requests into related groups, and the separability relation models gaps between such bunches. On the other hand, the visibility relation models a writer-reader dependency, similarly to the push-pop dependency in this paper. We have also applied the approach to queues, where it derived a mildly streamlined variant of the queue axioms of Henzinger et al. [10, 2], and to locks, including readers-writers locks. In the future, we plan to study if this pattern applies to other concurrent data structures (e.g., memory snapshots, trees, lists, sets, etc.).

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