



Unifying Asynchronous Logics for Hyperproperties

Alberto Bombardelli  

FBK, Povo TN, Italy

Laura Bozzelli 

University of Napoli “Federico II”, Napoli, Italy

César Sánchez  

IMDEA Software Institute, Madrid, Spain

Stefano Tonetta  

FBK, Povo TN, Italy

Abstract

We introduce and investigate a powerful hyper logical framework in the linear-time setting that we call *generalized HyperLTL with stuttering and contexts* (GHyperLTL_{S+C} for short). GHyperLTL_{S+C} unifies the asynchronous extensions of HyperLTL called HyperLTL_S and HyperLTL_C, and the well-known extension KLTL of LTL with knowledge modalities under both the synchronous and asynchronous perfect recall semantics. As a main contribution, we identify a meaningful fragment of GHyperLTL_{S+C}, that we call *simple GHyperLTL_{S+C}*, with a decidable model-checking problem, which is more expressive than HyperLTL and known fragments of asynchronous extensions of HyperLTL with a decidable model-checking problem. Simple GHyperLTL_{S+C} subsumes KLTL under the synchronous semantics and the one-agent fragment of KLTL under the asynchronous semantics and to the best of our knowledge, it represents the unique hyper logic with a decidable model-checking problem which can express powerful non-regular trace properties when interpreted on singleton sets of traces. We justify the relevance of simple GHyperLTL_{S+C} by showing that it can express diagnosability properties, interesting classes of information-flow security policies, both in the synchronous and asynchronous settings, and bounded termination (more in general, global promptness in the style of Prompt LTL).

2012 ACM Subject Classification Theory of computation → Logic and verification

Keywords and phrases Asynchronous hyperproperties, Temporal logics for hyperproperties, Expressiveness, Decidability, Model checking

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2024.15

1 Introduction

Temporal logics [28] play a fundamental role in the formal verification of the dynamic behaviour of complex reactive systems. Classic *regular* temporal logics such as LTL, CTL, and CTL* [31, 12] are suited for the specification of *trace properties* which describe the ordering of events along individual execution traces of a system. In the last 15 years, a novel specification paradigm has been introduced that generalizes traditional regular trace properties by properties of sets of traces, the so called *hyperproperties* [9]. Hyperproperties relate distinct traces and are useful to formalize a wide range of properties of prime interest which go, in general, beyond regular properties and cannot be expressed in standard regular temporal logics. A relevant example concerns information-flow security policies like noninterference [17, 29] and observational determinism [39] which compare observations made by an external low-security agent along traces resulting from different values of not directly observable inputs. Other examples include bounded termination of programs, diagnosability of critical systems (which amounts to checking whether the available sensor information is sufficient to infer the presence of faults on the hidden behaviour of the system) [33, 4, 3], and epistemic properties describing the knowledge of agents in distributed systems [22, 35, 21].



© Alberto Bombardelli, Laura Bozzelli, César Sánchez and Stefano Tonetta;
licensed under Creative Commons License CC-BY 4.0

44th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2024).

Editors: Siddharth Barman and Sławomir Lasota; Article No. 15; pp. 15:1–15:37



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In the context of model checking of finite-state reactive systems, many temporal logics for hyperproperties have been proposed [11, 8, 5, 32, 14, 10, 19] for which model checking is decidable, including HyperLTL [8], HyperCTL* [8], HyperQPTL [32, 10], and HyperPDL- Δ [19] which extend LTL, CTL*, QPTL [34], and PDL [16], respectively, by explicit first-order quantification over traces and trace variables to refer to multiple traces at the same time. The semantics of all these logics is *synchronous*: the temporal modalities are evaluated by a lockstepwise traversal of all the traces assigned to the quantified trace variables. Other approaches for the formalization of synchronous hyper logics are either based on hyper variants of monadic second-order logic over traces or trees [10], or the adoption of a *team semantics* for standard temporal logics, in particular, LTL [23, 27, 38]. For the first approach in the linear-time setting, we recall the logic S1S[E] [10] (and its first-order fragment FO[\langle, E] [15]) which syntactically extends monadic second-order logic of one successor S1S with the *equal-level predicate* E, which relates the same time point on different traces. More recently, an extension of HyperLTL with second-order quantification over traces has been introduced [2] which allows to express common knowledge in multi-agent distributed systems. Like S1S[E], model checking of this extension of HyperLTL is highly undecidable [2].

Hyper logics supporting asynchronous features have been introduced recently [20, 1, 6]. These logics allow to relate traces at distinct time points which can be arbitrarily far from each other. Asynchronicity is ubiquitous in many real-world systems, for example, in multithreaded environments in which threads are not scheduled lockstepwise, and traces associated with distinct threads progress with different speed. Asynchronous hyperproperties are also useful in information-flow security and diagnosability settings where an observer cannot distinguish consecutive time points along an execution having the same observations. This requires to match asynchronously sequences of observations along distinct execution traces. The first systematic study of asynchronous hyperproperties was done by Gutsfeld et al. [20], who introduced the temporal fixpoint calculus H_μ and its automata-theoretic counterpart for expressing such properties in the linear-time setting.

More recently, three temporal logics [1, 6] which syntactically extend HyperLTL have been introduced for expressing asynchronous hyperproperties: *Asynchronous HyperLTL* (A-HyperLTL) [1] and *Stuttering HyperLTL* (HyperLTL_S) [6], both useful for asynchronous security analysis, and *Context HyperLTL* (HyperLTL_C) [6], useful for expressing hyper-bounded-time response requirements. The logic A-HyperLTL, which is expressively incomparable with both HyperLTL and HyperLTL_S [7], models asynchronicity by means of an additional quantification layer over the so called *trajectories* which control the relative speed at which traces progress by choosing at each instant which traces move and which traces stutter. On the other hand, the logic HyperLTL_S exploits relativized versions of the temporal modalities with respect to finite sets Γ of LTL formulas: these modalities are evaluated by a lockstepwise traversal of the sub-traces of the given traces which are obtained by removing “redundant” positions with respect to the pointwise evaluation of the LTL formulas in Γ . Finally, the logic HyperLTL_C is more expressive than HyperLTL and is not expressively subsumed by either A-HyperLTL or HyperLTL_S [7]. HyperLTL_C extends HyperLTL by unary modalities $\langle C \rangle$ parameterized by a non-empty subset C of trace variables—called the *context*—which restrict the evaluation of the temporal modalities to the traces associated with the variables in C . Note that the temporal modalities in HyperLTL_C are evaluated by a lockstepwise traversal of the traces assigned to the variables in the current context, and unlike HyperLTL, the current time points of these traces from which the evaluation starts are in general different. It is known that these three syntactical extensions of HyperLTL are less expressive than H_μ [7] and like H_μ , model checking the respective quantifier alternation-free fragments are already

undecidable [1, 6]. The works [1, 6] identify practical fragments of the logics A-HyperLTL and HyperLTL_S with a decidable model checking problem. In particular, we recall the so called *simple fragment* of HyperLTL_S [6], which is more expressive than HyperLTL [7] and can specify interesting security policies in both the asynchronous and synchronous settings.

Formalization of asynchronous hyperproperties in the *team semantics setting* following an approach similar to the *trajectory construct* of A-HyperLTL has been investigated in [18]. It is worth noting that unlike other hyper logics (including logics with team semantics) which only capture regular trace properties when interpreted on singleton sets of traces, the logics HyperLTL_C, A-HyperLTL, and H_μ can express non-regular trace properties [7].

Our contribution. Specifications in HyperLTL and in the known asynchronous extensions of HyperLTL, whose most expressive representative is H_μ [20], consist of a prefix of trace quantifiers followed by a quantifier-free formula which expresses temporal requirements on a fixed number of traces. Thus, these hyper logics lack mechanisms to relate directly an unbounded number of traces, which are required for example to express bounded termination or diagnosability properties [33, 4, 3]. This ability is partially supported by temporal logics with team semantics [23, 27, 38] and extensions of temporal logics with the knowledge modalities of epistemic logic [13], which relate computations whose histories are observationally equivalent for a given agent. In this paper, we introduce and investigate a hyper logical framework in the linear-time setting which unifies two known asynchronous extensions of HyperLTL and the well-known extension KLTL [22] of LTL with knowledge modalities under both the synchronous and asynchronous perfect recall semantics (where an agent remembers the whole sequence of its observations). The novel logic, that we call *generalized HyperLTL with stuttering and contexts* (GHyperLTL_{S+C} for short), merges HyperLTL_S and HyperLTL_C and adds two new natural modeling facilities: past temporal modalities for asynchronous hyperproperties and general trace quantification where trace quantifiers can occur in the scope of temporal modalities. Past temporal modalities used in combination with context modalities provide a powerful mechanism to compare histories of computations at distinct time points. Moreover, unrestricted trace quantification allows to relate an unbounded number of traces.

As a main contribution, we identify a meaningful fragment of GHyperLTL_{S+C} with a decidable model-checking problem, that we call *simple GHyperLTL_{S+C}*. This fragment is obtained from GHyperLTL_{S+C} by carefully imposing restrictions on the use of the stuttering and context modalities. Simple GHyperLTL_{S+C} allows quantification over arbitrary *pointed* traces (i.e., traces plus time points) in the style of FO[<,E] [15], it is more expressive than the simple fragment of HyperLTL_S [6], and it is expressively incomparable with full HyperLTL_S and S1S[E]. Moreover, this fragment subsumes both KLTL under the synchronous semantics and the one-agent fragment of KLTL under the asynchronous semantics. In fact, simple GHyperLTL_{S+C} can be seen as a very large fragment of GHyperLTL_{S+C} with a decidable model checking problem which (1) strictly subsumes HyperLTL and the simple fragment of HyperLTL_S, (2) is closed under Boolean connectives, and (3) allows an unrestricted nesting of temporal modalities. We justify the relevance of simple GHyperLTL_{S+C} by showing that it can express diagnosability properties, interesting classes of information-flow security policies, both in the synchronous and asynchronous settings, and bounded termination (more in general, global promptness in the style of Prompt LTL [24]). To the best of our knowledge, simple GHyperLTL_{S+C} represents the unique hyper logic with a decidable model-checking problem which can express powerful non-regular trace properties when interpreted over singleton sets of traces. Due to lack of space, some proofs are omitted and are given in the Appendix.

2 Background

We denote by \mathbb{N} the set of natural numbers. Given $i, j \in \mathbb{N}$, we write $[i, j]$ for the set of natural numbers h such that $i \leq h \leq j$, we use $[i, j)$ for the set $[i, j] \setminus \{j\}$, we use $(i, j]$ for the set $[i, j] \setminus \{i\}$, and $[i, \infty)$ for the set of natural numbers h such that $h \geq i$. Given a word w over some alphabet Σ , $|w|$ is the length of w ($|w| = \infty$ if w is infinite). For each $0 \leq i < |w|$, $w(i)$ is the $(i + 1)^{th}$ symbol of w , w^i is the suffix of w from position i , that is, the word $w(i)w(i + 1) \dots$, and $w[0, i]$ is the prefix of w that ends at position i .

We fix a finite set AP of atomic propositions. A *trace* is an infinite word over 2^{AP} , while a *finite trace* is a nonempty finite word over 2^{AP} . A *pointed trace* is a pair (σ, i) consisting of a trace σ and a position (timestamp) $i \in \mathbb{N}$ along σ .

Kripke structures. We define the dynamic behaviour of reactive systems by *Kripke structures* $\mathcal{K} = \langle S, S_0, E, \text{Lab} \rangle$ over a finite set AP of atomic propositions, where S is a set of states, $S_0 \subseteq S$ is the set of initial states, $E \subseteq S \times S$ is a transition relation which is total in the first argument (i.e., for each $s \in S$ there is $s' \in S$ with $(s, s') \in E$), and $\text{Lab} : S \rightarrow 2^{\text{AP}}$ is a labeling map assigning to each state s the set of propositions holding at s . The Kripke structure \mathcal{K} is finite if S is finite. A *path* π of \mathcal{K} is an infinite word $\pi = s_0, s_1, \dots$ over S such that $s_0 \in S_0$ and for all $i \geq 0$, $(s_i, s_{i+1}) \in E$. The path $\pi = s_0, s_1, \dots$ induces the trace $\text{Lab}(s_0)\text{Lab}(s_1) \dots$. A *trace of \mathcal{K}* is a trace induced by some path of \mathcal{K} . We denote by $\mathcal{L}(\mathcal{K})$ the set of traces of \mathcal{K} . A *finite path* of \mathcal{K} is a non-empty infix of some path of \mathcal{K} . We also consider *fair finite Kripke structures* (\mathcal{K}, F) , that is, finite Kripke structures \mathcal{K} equipped with a subset F of \mathcal{K} -states. A path π of \mathcal{K} is *F-fair* if π visits infinitely many times some state in F . We denote by $\mathcal{L}(\mathcal{K}, F)$ the set of traces of \mathcal{K} associated with the *F-fair* paths of \mathcal{K} .

Standard LTL with past (PLTL for short) [31]. Formulas ψ of PLTL over the given finite set AP of atomic propositions are defined by the following grammar:

$$\psi ::= \top \mid p \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\psi \mid \mathbf{Y}\psi \mid \psi \mathbf{U} \psi \mid \psi \mathbf{S} \psi$$

where $p \in \text{AP}$, \mathbf{X} and \mathbf{U} are the *next* and *until* temporal modalities respectively, and \mathbf{Y} (*previous* or *yesterday*) and \mathbf{S} (*since*) are their past counterparts. LTL is the fragment of PLTL that does not contain the past temporal modalities \mathbf{Y} and \mathbf{S} . We also use the following abbreviations: $\mathbf{F}\psi := \top \mathbf{U} \psi$ (*eventually*), $\mathbf{O}\psi := \top \mathbf{S} \psi$ (*past eventually* or *once*), and their duals $\mathbf{G}\psi := \neg \mathbf{F} \neg\psi$ (*always*) and $\mathbf{H}\psi := \neg \mathbf{O} \neg\psi$ (*past always* or *historically*).

The semantics of PLTL is defined over pointed traces (σ, i) . The satisfaction relation $(\sigma, i) \models \psi$, that defines whether formula ψ holds at position i along σ , is inductively defined as follows (we omit the semantics for the Boolean connectives which is standard):

$$\begin{aligned} (\sigma, i) \models p & \Leftrightarrow p \in \sigma(i) \\ (\sigma, i) \models \mathbf{X}\psi & \Leftrightarrow (\sigma, i + 1) \models \psi \\ (\sigma, i) \models \mathbf{Y}\psi & \Leftrightarrow i > 0 \text{ and } (\sigma, i - 1) \models \psi \\ (\sigma, i) \models \psi_1 \mathbf{U} \psi_2 & \Leftrightarrow \text{for some } j \geq i : (\sigma, j) \models \psi_2 \text{ and } (\sigma, k) \models \psi_1 \text{ for all } i \leq k < j \\ (\sigma, i) \models \psi_1 \mathbf{S} \psi_2 & \Leftrightarrow \text{for some } j \leq i : (\sigma, j) \models \psi_2 \text{ and } (\sigma, k) \models \psi_1 \text{ for all } j < k \leq i \end{aligned}$$

A trace σ is a *model* of ψ , written $\sigma \models \psi$, whenever $(\sigma, 0) \models \psi$.

The logic HyperLTL [8]. The syntax of HyperLTL formulas φ over the given finite set AP of atomic propositions and a finite set VAR of trace variables is as follows:

$$\varphi := \exists x. \varphi \mid \forall x. \varphi \mid \psi \quad \psi := \top \mid p[x] \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\psi \mid \psi \mathbf{U} \psi$$

where $p \in \text{AP}$, $x \in \text{VAR}$, and $\exists x$ and $\forall x$ are the *hyper* existential and universal trace quantifiers for variable x , respectively, which allow relating different traces of the given set

of traces. Note that a HyperLTL formula consists of a prefix of traces quantifiers followed by a quantifier-free formula, where the latter corresponds to an LTL formula whose atomic propositions p are replaced with x -relativized versions $p[x]$. Intuitively, $p[x]$ asserts that p holds at the pointed trace assigned to variable x . A *sentence* is a formula where each relativized proposition $p[x]$ occurs in the scope of trace quantifier $\exists x$ or $\forall x$.

In order to define the semantics of HyperLTL, we need additional definitions. The *successor* $\text{succ}(\sigma, i)$ of a pointed trace (σ, i) is the pointed trace $(\sigma, i + 1)$, which captures the standard local successor of a position along a trace.

Given a set of traces \mathcal{L} , a (*pointed*) *trace assignment* over \mathcal{L} is a partial mapping $\Pi : \text{VAR} \rightarrow \mathcal{L} \times \mathbb{N}$ assigning to each trace variable x —where Π is defined—a pointed trace (σ, i) such that $\sigma \in \mathcal{L}$. We use $\text{Dom}(\Pi)$ to refer to the trace variables for which Π is defined. The *successor* $\text{succ}(\Pi)$ of Π is the trace assignment over \mathcal{L} having domain $\text{Dom}(\Pi)$ such that $\text{succ}(\Pi)(x) = \text{succ}(\Pi(x))$ for each $x \in \text{Dom}(\Pi)$. For each $i \geq 0$, we use succ^i for the function obtained by i applications of the function succ : $\text{succ}^0(\Pi) := \Pi$ and $\text{succ}^{i+1}(\Pi) := \text{succ}(\text{succ}^i(\Pi))$.

Given $x \in \text{VAR}$ and a pointed trace (σ, i) with $\sigma \in \mathcal{L}$, we denote by $\Pi[x \mapsto (\sigma, i)]$ the trace assignment that is identical to Π besides for x , which is mapped to (σ, i) .

Given a formula φ , a set of traces \mathcal{L} , and a trace assignment Π over \mathcal{L} such that $\text{Dom}(\Pi)$ contains all the trace variables occurring free in φ , the satisfaction relation $\Pi \models_{\mathcal{L}} \varphi$ is inductively defined as follows (we again omit the semantics of the Boolean connectives):

$$\begin{aligned} \Pi \models_{\mathcal{L}} p[x] &\Leftrightarrow \Pi(x) = (\sigma, i) \text{ and } p \in \sigma(i) \\ \Pi \models_{\mathcal{L}} \exists x. \varphi &\Leftrightarrow \text{for some } \sigma \in \mathcal{L}, \Pi[x \mapsto (\sigma, 0)] \models_{\mathcal{L}} \varphi \\ \Pi \models_{\mathcal{L}} \forall x. \varphi &\Leftrightarrow \text{for all } \sigma \in \mathcal{L}, \Pi[x \mapsto (\sigma, 0)] \models_{\mathcal{L}} \varphi \\ \Pi \models_{\mathcal{L}} \mathbf{X}\psi &\Leftrightarrow \text{succ}(\Pi) \models \psi \\ \Pi \models_{\mathcal{L}} \psi_1 \mathbf{U} \psi_2 &\Leftrightarrow \text{for some } i \geq 0 : \text{succ}^i(\Pi) \models_{\mathcal{L}} \psi_2 \text{ and } \text{succ}^j(\Pi) \models_{\mathcal{L}} \psi_1 \text{ for all } 0 \leq j < i \end{aligned}$$

Note that trace quantification ranges over *initial* pointed traces $(\sigma, 0)$ over \mathcal{L} (the timestamp is 0). As an example, the sentence $\forall x_1. \forall x_2. \bigwedge_{p \in \text{AP}} \mathbf{G}(p[x_1] \leftrightarrow p[x_2])$ captures the sets of traces which are singletons.

For a sentence φ and a set of traces \mathcal{L} , \mathcal{L} is a *model* of φ , written $\mathcal{L} \models \varphi$, if $\Pi_{\emptyset} \models_{\mathcal{L}} \varphi$ where Π_{\emptyset} is the trace assignment with empty domain.

3 Unifying Framework for Asynchronous Extensions of HyperLTL

In this section, we introduce a novel logical framework for specifying both asynchronous and synchronous linear-time hyperproperties which unifies two known more expressive extensions of HyperLTL [8], namely *Stuttering HyperLTL* (HyperLTL_S for short) [6] and *Context HyperLTL* (HyperLTL_C for short) [6]. The proposed hyper logic, that we call *generalized HyperLTL with stuttering and contexts* (GHyperLTL_{S+C} for short), merges HyperLTL_S and HyperLTL_C and adds two new features: past temporal modalities for asynchronous/synchronous hyperproperties and general trace quantification where trace quantifiers can occur in the scope of temporal modalities. Since model checking of the logics HyperLTL_S and HyperLTL_C is already undecidable [6], we also consider a meaningful fragment of GHyperLTL_{S+C} which is strictly more expressive than the known *simple fragment* of HyperLTL_S [6]. Our fragment is able to express relevant classes of hyperproperties and, as we show in Section 4, its model checking problem is decidable.

3.1 PLTL-Relativized Stuttering and Context Modalities

Classically, a trace is stutter-free if there are no consecutive positions having the same propositional valuation unless the valuation is repeated ad-infinitum. We can associate to each trace a unique stutter-free trace by removing “redundant” positions. The logic HyperLTL_5 [6] generalizes these notions with respect to the pointwise evaluation of a finite set of LTL formulas. Here, we consider LTL with past (PLTL).

► **Definition 3.1** (PLTL stutter factorization [6]). *Let Γ be a finite set of PLTL formulas and σ a trace. The Γ -stutter factorization of σ is the unique increasing sequence of positions $\{i_k\}_{k \in [0, m_\infty]}$ for some $m_\infty \in \mathbb{N} \cup \{\infty\}$ such that the following holds for all $j < m_\infty$:*

- $i_0 = 0$ and $i_j < i_{j+1}$;
- for each $\theta \in \Gamma$, the truth value of θ along the segment $[i_j, i_{j+1})$ does not change, that is, for all $h, k \in [i_j, i_{j+1})$, $(\sigma, h) \models \theta$ iff $(\sigma, k) \models \theta$, and the same holds for the infinite segment $[m_\infty, \infty)$ in case $m_\infty \neq \infty$;
- the truth value of some formula in Γ changes along adjacent segments, that is, for some $\theta \in \Gamma$ (depending on j), $(\sigma, i_j) \models \theta$ iff $(\sigma, i_{j+1}) \not\models \theta$.

Thus, the Γ -stutter factorization $\{i_k\}_{k \in [0, m_\infty]}$ of σ partitions the trace in adjacent non-empty segments such that the valuation of formulas in Γ does not change within a segment, and changes in moving from a segment to the adjacent ones. This factorization induces in a natural way a trace obtained by selecting the first positions of the finite segments and all the positions of the unique tail infinite segment, if any. These positions form an infinite increasing sequence $\{\ell_k\}_{k \in \mathbb{N}}$ called (Γ, ω) -stutter factorization of σ , where:

$$\ell_0, \ell_1, \dots := \begin{cases} i_0, i_1, \dots & \text{if } m_\infty = \infty \\ i_0, i_1, \dots, i_{m_\infty}, i_{m_\infty} + 1, \dots & \text{otherwise} \end{cases}$$

The Γ -stutter trace $\text{stfr}_\Gamma(\sigma)$ of σ (see [6]) is defined as follows: $\text{stfr}_\Gamma(\sigma) := \sigma(\ell_0)\sigma(\ell_1)\dots$. Note that for $\Gamma = \emptyset$, $\text{stfr}_\Gamma(\sigma) = \sigma$. A trace σ is Γ -stutter free if it coincides with its Γ -stutter trace, i.e. $\text{stfr}_\Gamma(\sigma) = \sigma$.

As an example, assume that $\text{AP} = \{p, q, r\}$ and let $\Gamma = \{p \text{ U } q\}$. Given $h, k \geq 1$, let $\sigma_{h,k}$ be the trace $\sigma_{h,k} = p^h q^k r^\omega$. These traces have the same Γ -stutter trace given by pr^ω .

The semantics of the Γ -relativized temporal modalities in HyperLTL_5 is based on the notion of Γ -successor $\text{succ}_\Gamma(\sigma, i)$ of a pointed trace (σ, i) [6]: $\text{succ}_\Gamma(\sigma, i)$ is the pointed trace (σ, ℓ) where ℓ is the smallest position ℓ_j in the (Γ, ω) -stutter factorization $\{\ell_k\}_{k \in \mathbb{N}}$ of σ which is greater than i . Note that for $\Gamma = \emptyset$, $\text{succ}_\emptyset(\sigma, i) = \text{succ}(\sigma, i) = (\sigma, i+1)$. Hence, \emptyset -relativized temporal modalities in HyperLTL_5 correspond to the temporal modalities of HyperLTL .

In this paper we extend HyperLTL_5 with past temporal modalities, so that we introduce the past counterpart of the successor function. The Γ -predecessor $\text{pred}_\Gamma(\sigma, i)$ of a pointed trace (σ, i) is undefined if $i = 0$ (written $\text{pred}_\Gamma(\sigma, i) = \text{und}$); otherwise, $\text{pred}_\Gamma(\sigma, i)$ is the pointed trace (σ, ℓ) where ℓ is the greatest position ℓ_j in the (Γ, ω) -stutter factorization $\{\ell_k\}_{k \in \mathbb{N}}$ of σ which is smaller than i (since $\ell_0 = 0$ such an ℓ_j exists). Note that for $\Gamma = \emptyset$, $\text{pred}_\emptyset(\sigma, i)$ captures the standard local predecessor of a position along a trace.

Successors and predecessors of trace assignments. We now define a generalization of the successor $\text{succ}(\Pi)$ of a trace assignment Π in HyperLTL . This generalization is based on the notion of Γ -successor $\text{succ}_\Gamma(\sigma, i)$ of a pointed trace (σ, i) and also takes into account the context modalities $\langle C \rangle$ of HyperLTL_C [6], where a *context* C is a non-empty subset of VAR . Intuitively, modality $\langle C \rangle$ allows reasoning over a subset of the traces assigned to the variables in the formula, by restricting the temporal progress to those traces.

Formally, let Π be a trace assignment over some set of traces \mathcal{L} , Γ be a finite set of PLTL formulas, and C be a context. The (Γ, C) -successor of Π , denoted by $\text{succ}_{(\Gamma, C)}(\Pi)$, is the trace assignment over \mathcal{L} having domain $\text{Dom}(\Pi)$, and defined as follows for each $x \in \text{Dom}(\Pi)$:

$$\text{succ}_{(\Gamma, C)}(\Pi)(x) := \begin{cases} \text{succ}_{\Gamma}(\Pi(x)) & \text{if } x \in C \\ \Pi(x) & \text{otherwise} \end{cases}$$

Note that $\text{succ}_{(\emptyset, \text{VAR})}(\Pi) = \text{succ}(\Pi)$. Moreover, we define the (Γ, C) -predecessor $\text{pred}_{(\Gamma, C)}(\Pi)$ of Π as follows: $\text{pred}_{(\Gamma, C)}(\Pi)$ is *undefined*, written $\text{pred}_{(\Gamma, C)}(\Pi) = \mathbf{und}$, if there is $x \in \text{Dom}(\Pi)$ such that $\text{pred}_{\Gamma}(\Pi(x)) = \mathbf{und}$. Otherwise, $\text{pred}_{(\Gamma, C)}(\Pi)$ is the trace assignment over \mathcal{L} having domain $\text{Dom}(\Pi)$, and defined as follows for each $x \in \text{Dom}(\Pi)$:

$$\text{pred}_{(\Gamma, C)}(\Pi)(x) := \begin{cases} \text{pred}_{\Gamma}(\Pi(x)) & \text{if } x \in C \\ \Pi(x) & \text{otherwise} \end{cases}$$

Finally, for each $i \geq 0$, we define the i^{th} application $\text{succ}_{(\Gamma, C)}^i$ of $\text{succ}_{(\Gamma, C)}$ and the i^{th} application $\text{pred}_{(\Gamma, C)}^i$ of $\text{pred}_{(\Gamma, C)}$ as follows, where $\text{pred}_{(\Gamma, C)}(\mathbf{und}) := \mathbf{und}$:

- $\text{succ}_{(\Gamma, C)}^0(\Pi) := \Pi$ and $\text{succ}_{(\Gamma, C)}^{i+1}(\Pi) := \text{succ}_{(\Gamma, C)}(\text{succ}_{(\Gamma, C)}^i(\Pi))$.
- $\text{pred}_{(\Gamma, C)}^0(\Pi) := \Pi$ and $\text{pred}_{(\Gamma, C)}^{i+1}(\Pi) := \text{pred}_{(\Gamma, C)}(\text{pred}_{(\Gamma, C)}^i(\Pi))$.

3.2 Generalized HyperLTL with Stuttering and Contexts

We introduce now the novel logic GHyperLTL_{S+C} . GHyperLTL_{S+C} formulas φ over AP and a finite set VAR of trace variables are defined by the following syntax:

$$\varphi := \top \mid p[x] \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x. \varphi \mid \langle C \rangle \varphi \mid \mathbf{X}_{\Gamma} \varphi \mid \mathbf{Y}_{\Gamma} \varphi \mid \varphi \mathbf{U}_{\Gamma} \varphi \mid \varphi \mathbf{S}_{\Gamma} \varphi$$

where $p \in \text{AP}$, $x \in \text{VAR}$, $\langle C \rangle$ is the context modality with $\emptyset \neq C \subseteq \text{VAR}$, Γ is a finite set of PLTL formulas, and \mathbf{X}_{Γ} , \mathbf{Y}_{Γ} , \mathbf{U}_{Γ} and \mathbf{S}_{Γ} are the stutter-relativized versions of the PLTL temporal modalities. Intuitively, the context modality $\langle C \rangle$ restricts the evaluation of the temporal modalities to the traces associated with the variables in C , while the temporal modalities \mathbf{X}_{Γ} , \mathbf{Y}_{Γ} , \mathbf{U}_{Γ} and \mathbf{S}_{Γ} are evaluated by a lockstepwise traversal of the Γ -stutter traces associated to the traces assigned to the variables in the current context C . Note that the hyper universal quantifier $\forall x$ can be introduced as an abbreviation: $\forall x. \varphi \equiv \neg \exists x. \neg \varphi$. For a variable x , we write $\langle x \rangle$ instead of $\langle \{x\} \rangle$. Moreover, we write \mathbf{X} , \mathbf{Y} , \mathbf{U} and \mathbf{S} instead of \mathbf{X}_{\emptyset} , \mathbf{Y}_{\emptyset} , \mathbf{U}_{\emptyset} and \mathbf{S}_{\emptyset} , respectively. Furthermore, for a PLTL formula ψ and a variable x , $\psi[x]$ is the formula obtained from ψ by replacing each proposition p with its x -version $p[x]$. A *sentence* is a formula where each relativized proposition $p[x]$ occurs in the scope of trace quantifier $\exists x$ or $\forall x$, and each temporal modality occurs in the scope of a trace quantifier.

The known logics HyperLTL_S, HyperLTL_C, and simple HyperLTL_S. A formula φ of GHyperLTL_{S+C} is in *prenex* form if it is of the form $\text{Q}_1 x_1. \dots \text{Q}_n x_n. \psi$ where ψ is quantifier-free and $\text{Q}_i \in \{\exists, \forall\}$ for all $i \in [1, n]$. The logics HyperLTL_S and HyperLTL_C introduced in [6] correspond to syntactical fragments of GHyperLTL_{S+C} where the formulas are in prenex form and past temporal modalities are not used. Moreover, in HyperLTL_S , the context modalities are not allowed, while in HyperLTL_C , the subscript Γ of every temporal modality must be the empty set. Note that in HyperLTL [8], both the context modalities and the temporal modalities where the subscript Γ is not empty are disallowed. Finally, we recall the *simple* fragment of HyperLTL_S [6], which is more expressive than HyperLTL and is parameterized by a finite set Γ of LTL formulas. The *quantifier-free* formulas of simple HyperLTL_S for the

15:8 Unifying Asynchronous Logics for Hyperproperties

parameter Γ are defined as Boolean combinations of formulas of the form $\psi[x]$, where ψ is an LTL formula, and formulas ψ_Γ defined by the following grammar:

$$\psi_\Gamma := \top \mid p[x] \mid \neg\psi_\Gamma \mid \psi_\Gamma \vee \psi_\Gamma \mid \mathbf{X}_\Gamma\psi_\Gamma \mid \psi_\Gamma \mathbf{U}_\Gamma\psi_\Gamma$$

Semantics of GHYperLTL_{S+C}. Given a formula φ , a set of traces \mathcal{L} , a trace assignment Π over \mathcal{L} such that $\text{Dom}(\Pi)$ contains all the trace variables occurring free in φ , and a context $C \subseteq \text{VAR}$, the satisfaction relation $(\Pi, C) \models_{\mathcal{L}} \varphi$, meaning that the assignment Π over \mathcal{L} satisfies φ under the context C , is inductively defined as follows (we again omit the semantics of the Boolean connectives):

$$\begin{aligned} (\Pi, C) \models_{\mathcal{L}} p[x] &\Leftrightarrow \Pi(x) = (\sigma, i) \text{ and } p \in \sigma(i) \\ (\Pi, C) \models_{\mathcal{L}} \exists x. \varphi &\Leftrightarrow \text{for some } \sigma \in \mathcal{L}, (\Pi[x \mapsto (\sigma, 0)], C) \models_{\mathcal{L}} \varphi \\ (\Pi, C) \models_{\mathcal{L}} \langle C' \rangle \varphi &\Leftrightarrow (\Pi, C') \models_{\mathcal{L}} \varphi \\ (\Pi, C) \models_{\mathcal{L}} \mathbf{X}_\Gamma \varphi &\Leftrightarrow (\text{succ}_{(\Gamma, C)}(\Pi), C) \models \varphi \\ (\Pi, C) \models_{\mathcal{L}} \mathbf{Y}_\Gamma \varphi &\Leftrightarrow \text{pred}_{(\Gamma, C)}(\Pi) \neq \mathbf{und} \text{ and } (\text{pred}_{(\Gamma, C)}(\Pi), C) \models_{\mathcal{L}} \varphi \\ (\Pi, C) \models_{\mathcal{L}} \varphi_1 \mathbf{U}_\Gamma \varphi_2 &\Leftrightarrow \text{for some } i \geq 0: (\text{succ}_{(\Gamma, C)}^i(\Pi), C) \models_{\mathcal{L}} \varphi_2 \text{ and} \\ &\quad (\text{succ}_{(\Gamma, C)}^j(\Pi), C) \models_{\mathcal{L}} \varphi_1 \text{ for all } 0 \leq j < i, \\ (\Pi, C) \models_{\mathcal{L}} \varphi_1 \mathbf{S}_\Gamma \varphi_2 &\Leftrightarrow \text{for some } i \geq 0 \text{ such that } \text{pred}_{(\Gamma, C)}^i(\Pi) \neq \mathbf{und}: (\text{pred}_{(\Gamma, C)}^i(\Pi), C) \models_{\mathcal{L}} \varphi_2 \\ &\quad \text{and } (\text{pred}_{(\Gamma, C)}^j(\Pi), C) \models_{\mathcal{L}} \varphi_1 \text{ for all } 0 \leq j < i \end{aligned}$$

For a sentence φ and a set of traces \mathcal{L} , \mathcal{L} is a *model* of φ , written $\mathcal{L} \models \varphi$, if $(\Pi_\emptyset, \text{VAR}) \models_{\mathcal{L}} \varphi$ where Π_\emptyset is the trace assignment with empty domain.

Fair model checking and standard model checking. For a fragment \mathcal{F} of GHYperLTL_{S+C}, the *fair model checking problem* for \mathcal{F} consists on deciding, given a fair finite Kripke structure (\mathcal{K}, F) and a sentence φ of \mathcal{F} , whether $\mathcal{L}(\mathcal{K}, F) \models \varphi$. The previous problem is simply called *model checking problem* whenever F coincides with the set of \mathcal{K} -states. We consider fair model checking just for technical convenience. For the decidable fragment of GHYperLTL_{S+C} introduced in Section 3.3, we will obtain the same complexity bounds for both fair model checking and standard model checking (see Section 4).

3.3 The Simple Fragment of GHYperLTL_{S+C}

We introduce now a fragment of GHYperLTL_{S+C}, that we call *simple* GHYperLTL_{S+C}, which syntactically subsumes the simple fragment of HyperLTL_S [6].

In order to define the syntax of simple GHYperLTL_{S+C}, we first consider some shorthands, obtained by a restricted use of the context modalities. The *pointed existential quantifier* $\exists^P x$ and the *pointed universal quantifier* $\forall^P x$ are defined as follows: $\exists^P x. \varphi := \exists x. \langle x \rangle \mathbf{F} \langle \text{VAR} \rangle \varphi$ and $\forall^P x. \varphi ::= \neg \exists^P x. \neg \varphi$. Thus the pointed quantifiers quantify on arbitrary pointed traces over the given set of traces and set the global context for the given operand. Formally, $(\Pi, C) \models_{\mathcal{L}} \exists^P x. \varphi$ if for some pointed trace (σ, i) with $\sigma \in \mathcal{L}$, $(\Pi[x \mapsto (\sigma, i)], \text{VAR}) \models_{\mathcal{L}} \varphi$.

For example, the sentence $\exists x_1. \exists^P x_2. (\bigwedge_{p \in \text{AP}} \mathbf{G}(p[x_1] \leftrightarrow p[x_2]) \wedge \langle x_2 \rangle \mathbf{Y} \top)$ asserts that there are two traces σ_1 and σ_2 in the given model s.t. some *proper* suffix of σ_2 coincides with σ_1 .

Simple GHYperLTL_{S+C} is parameterized by a finite set Γ of PLTL formulas. The set of formulas φ_Γ in the Γ -fragment is defined as follows:

$$\varphi_\Gamma := \top \mid \langle x \rangle \psi[x] \mid \neg\varphi_\Gamma \mid \varphi_\Gamma \vee \varphi_\Gamma \mid \exists^P x. \varphi_\Gamma \mid \mathbf{X}_\Gamma \varphi_\Gamma \mid \mathbf{Y}_\Gamma \varphi_\Gamma \mid \varphi_\Gamma \mathbf{U}_\Gamma \varphi_\Gamma \mid \varphi_\Gamma \mathbf{S}_\Gamma \varphi_\Gamma$$

where ψ is a PLTL formula. Note that $\exists x. \varphi$ can be expressed as $\exists^P x. (\varphi \wedge \langle x \rangle \neg \mathbf{Y} \top)$. SHYperLTL_{S+C} ^{Γ} is the class of formulas obtained with the syntax above for a given Γ . Simple GHYperLTL_{S+C} is the union SHYperLTL_{S+C} ^{Γ} for all Γ . We say that a formula φ of simple

$\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$ is *singleton-free* if for each subformula $\langle x \rangle \psi[x]$ of φ , ψ is an atomic proposition. Evidently, for an atomic proposition p , $\langle x \rangle p[x]$ is equivalent to $p[x]$.

In Section 4, we will show that (fair) model checking of simple $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$ is decidable. Simple $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$ can be seen as a very large fragment of $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$ with a decidable model checking problem which subsumes the simple fragment of $\text{HyperLTL}_{\mathcal{S}}$, is closed under Boolean connectives, and allows an unrestricted nesting of temporal modalities. We conjecture (without proof) that this is the largest such sub-class of $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$ because:

1. Model checking of $\text{HyperLTL}_{\mathcal{S}}$ is already undecidable [6] for sentences whose relativized temporal modalities exploit two distinct sets of LTL formulas and, in particular, for two-variable quantifier alternation-free sentences of the form $\exists x_1. \exists x_2. (\varphi \wedge \mathbf{G}_{\Gamma} \psi)$, where ψ is a propositional formula, Γ is a nonempty set of propositions, and φ is a quantifier-free formula which use only the temporal modalities \mathbf{F}_{\emptyset} and \mathbf{G}_{\emptyset} .
2. Model checking of $\text{HyperLTL}_{\mathcal{C}}$ is undecidable [7] even for the fragment consisting of two-variable quantifier alternation-free sentences of the form $\exists x_1. \exists x_2. \psi_0 \wedge \langle x_2 \rangle \mathbf{F} \langle \{x_1, x_2\} \rangle \psi$, where ψ_0 and ψ are quantifier-free HyperLTL formulas (note that ψ_0 is evaluated in the global context $\langle \{x_1, x_2\} \rangle$).

The second undecidability result suggests to consider the extension of simple $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$ where singleton-context subformulas of the form $\langle x \rangle \psi[x]$ are replaced with *quantifier-free* $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$ formulas with multiple variables of the form $\langle x \rangle \xi$, where ξ only uses singleton contexts $\langle y \rangle$ and temporal modalities with subscript \emptyset . However, we can show that the resulting logic is not more expressive than simple $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$: a sentence in the considered extension can be translated into an equivalent simple $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$ sentence though with a non-elementary blowup (for details, see Appendix A.1).

3.4 Examples of Specifications in Simple $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$

We consider some relevant properties which can be expressed in simple $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$. Simple $\text{GHyperLTL}_{\mathcal{S}+\mathcal{C}}$ subsumes the simple fragment of $\text{HyperLTL}_{\mathcal{S}}$, and this fragment (as shown in [6]) can express relevant information-flow security properties for asynchronous frameworks such as distributed systems or cryptographic protocols. An example is the asynchronous variant of the *noninterference* property, as defined by Goguen and Meseguer [17], which asserts that the observations of low users (users accessing only to public information) do not change when all inputs of high users (users accessing secret information) are removed.

Observational Determinism.

An important information-flow property is observational determinism, which states that traces which have the same initial low inputs are indistinguishable to a low user. In an asynchronous setting, a user cannot infer that a transition occurred if consecutive observations remain unchanged. Thus, for instance, observational determinism with equivalence of traces up to stuttering (as formulated in [39]) can be captured by the following simple $\text{HyperLTL}_{\mathcal{S}}$ sentence (where LI is the set of propositions describing inputs of low users and LO is set of propositions describing outputs of low users):

$$\forall x. \forall y. \bigwedge_{p \in LI} (p[x] \leftrightarrow p[y]) \rightarrow \mathbf{G}_{LO} \bigwedge_{p \in LO} (p[x] \leftrightarrow p[y])$$

After-initialization Properties.

Simple GHyperLTL_{S+C} also allows to specify complex combinations of asynchronous and synchronous constraints. As an example, we consider the property [20] that for an HyperLTL sentence $\mathbf{Q}_1 x_1 \dots \mathbf{Q}_n x_n \cdot \psi(x_1, \dots, x_n)$, the quantifier-free formula $\psi(x_1, \dots, x_n)$ holds along the traces bound by variables x_1, \dots, x_n after an initialization phase. Note that this phase can take a different (and unbounded) number of steps on each trace. Let in be a proposition characterizing the initialization phase. The formula $PI(x) := \langle x \rangle (\neg \text{in}[x] \wedge (\neg \mathbf{Y}\top \vee \mathbf{Y}\text{in}[x]))$ is a simple GHyperLTL_{S+C} formula that asserts that for the pointed trace (σ, i) assigned to variable x , the position i is the first position of σ following the initialization phase. In other words, i is the first position at which $\neg \text{in}[]$ holds. Then, the previous requirement can be expressed in simple GHyperLTL_{S+C} as follows:

$$\mathbf{Q}_1 x_1 \dots \mathbf{Q}_n x_n \cdot (PI(x_1) \circ_1 \dots PI(x_n) \circ_n \psi(x_1, \dots, x_n))$$

where \circ_i is \wedge if $\mathbf{Q}_i = \exists$ and \circ_i is \rightarrow if $\mathbf{Q}_i = \forall$.

Global Promptness.

As another meaningful example, we consider global promptness (in the style of Prompt LTL [24]), where one needs to check that there is a uniform bound on the response time in all the traces of the system, that is, “*there is k such that for every trace, each request q is followed by a response p within k steps*”. Global promptness is expressible in simple GHyperLTL_{S+C} as follows:

$$\exists^P x \cdot (q[x] \wedge \forall^P y \cdot (q[y] \rightarrow (\neg p[x] \mathbf{U} p[y])))$$

The previous sentence asserts that there is request (x in the formula) that has the longest response. Note that y is quantified universally (so it can be instantiated to the same trace as x), and that the use of until in $(\neg p[x] \mathbf{U} p[y])$ implies that the response $p[y]$ eventually happens. Hence, all requests, including receive a response. Now, the pointed trace (σ_x, i) assigned to x is such that $\sigma_x(i)$ is a request ($q[x]$) and for every pointed trace (σ_y, j) in the model such that $\sigma_y(j)$ is a request ($q[y]$), it holds that (i) the request $\sigma_y(j)$ is followed by a response $\sigma_y(j+k)$ for some $k \geq 0$, and (ii) no response occurs in σ_x in the interval $[i, i+k)$. Therefore, the response time h for x is the smallest h such that $\sigma_x(i+h)$ is a response is a global bound on the response time.

Diagnosability.

We now show that simple GHyperLTL_{S+C} is also able to express *diagnosability* of systems [33, 4, 3] in a general asynchronous setting. In the diagnosis process, faults of a critical system (referred as the plant) are handled by a dedicated module (the *diagnoser*) which runs in parallel with the plant. The diagnoser analyzes the observable information from the plant—made available by predefined sensors—and triggers suitable alarms in correspondence to (typically unobservable) conditions of interest, called faults. An alarm condition specifies the relation (delay) between a given diagnosis condition and the raising of an alarm. A plant \mathcal{P} is *diagnosable* with respect to a given alarm condition α , if there is a diagnoser \mathcal{D} which satisfies α when \mathcal{D} runs in parallel with \mathcal{P} .

The given set of propositions AP is partitioned into a set of observable propositions Obs and a set of unobservable propositions Int . Two finite traces σ and σ' are *observationally equivalent* iff the projections of $\text{stfr}_{\text{Obs}}(\sigma \cdot P^\omega)$ and $\text{stfr}_{\text{Obs}}(\sigma' \cdot (P')^\omega)$ over Obs coincide, where

P is the last symbol of σ and similarly P' is the last symbol of σ' . Given a pointed trace (σ, i) , i is an *observation point* of σ if either $i = 0$, or $i > 0$ and $\sigma(i-1) \cap \text{Obs} \neq \sigma(i) \cap \text{Obs}$. Then a plant \mathcal{P} can be modeled as a finite Kripke structure $\langle S, S_0, E, Lab \rangle$, where E is partitioned into internal transitions (s, s') where $Lab(s) \cap \text{Obs} = Lab(s') \cap \text{Obs}$ and observable transitions where $Lab(s) \cap \text{Obs} \neq Lab(s') \cap \text{Obs}$. A diagnoser \mathcal{D} is modelled as a finite deterministic Kripke structure over $\text{AP}' \supseteq \text{Obs}$ (with $\text{AP}' \cap \text{Int} = \emptyset$). In the behavioural composition of the plant \mathcal{P} with \mathcal{D} , the diagnoser only reacts to the observable transitions of the plant, that is, every transition of the diagnoser is associated with an observable transition of the plant. Simple GHyperLTL_{S+C} can express diagnosability with *finite delay*, *bounded delay*, or *exact delay* as defined in [4, 3]. Here, we focus for simplicity on finite delay diagnosability. Consider a diagnosis condition specified by a PLTL formula β . A plant \mathcal{P} is *finite delay diagnosable* with respect to β whenever for every pointed trace (σ, i) of \mathcal{P} such that $(\sigma, i) \models \beta$, there exists an observation point $k \geq i$ of σ such that for all pointed traces (σ', k') of \mathcal{P} so that k' is an observation point of σ' and $\sigma[0, k]$ and $\sigma'[0, k']$ are observationally equivalent, it holds that $(\sigma', i') \models \beta$ for some $i' \leq k'$. Finite delay diagnosability w.r.t. β can be expressed in simple GHyperLTL_{S+C} as follows:

$$\forall^P x. \left(\langle x \rangle \beta[x] \rightarrow \mathbf{F}_{\text{Obs}} (\text{ObsPt}(x) \wedge \forall^P y. \{ (\text{ObsPt}(y) \wedge \theta_{\text{Obs}}(x, y)) \rightarrow \langle y \rangle \mathbf{O} \beta[y] \}) \right)$$

where

$$\begin{aligned} \theta_{\text{Obs}}(x, y) &:= \bigwedge_{p \in \text{Obs}} \mathbf{H}_{\text{Obs}} (p[x] \leftrightarrow p[y]) \wedge \mathbf{O}_{\text{Obs}} (\langle x \rangle \neg \mathbf{Y} \top \wedge \langle y \rangle \neg \mathbf{Y} \top) \\ \text{ObsPt}(x) &:= \langle x \rangle (\neg \mathbf{Y} \top \wedge \bigvee_{p \in \text{Obs}} (p[x] \leftrightarrow \neg \mathbf{Y} p[x])) \end{aligned}$$

Essentially $\text{ObsPt}(x)$ determines the observation points and θ_{Obs} captures that both traces have the same history of observations. The main formula establishes that if x detects a failure β then there is future observation point in x and for all other traces that are observationally equivalent to x have also detected β .

3.5 Expressiveness Issues

In this section, we present some results and conjectures about the expressiveness comparison among GHyperLTL_{S+C} (which subsumes HyperLTL_S and HyperLTL_C), its simple fragment and the logic HyperLTL_S . We also consider the logics for linear-time hyperproperties based on the equal-level predicate whose most powerful representative is $\text{S1S}[E]$. Recall that the first-order fragment $\text{FO}[\langle, E \rangle]$ of $\text{S1S}[E]$ is already strictly more expressive than HyperLTL [15] and, unlike $\text{S1S}[E]$, its model-checking problem is decidable [10]. Moreover, we show that GHyperLTL_{S+C} and its simple fragment represent a unifying framework in the linear-time setting for specifying both hyperproperties and the knowledge modalities of epistemic temporal logics under both the synchronous and asynchronous perfect recall semantics.

Our expressiveness results about linear-time hyper logics can be summarized as follows.

► **Theorem 3.2.** *The following hold:*

- GHyperLTL_{S+C} is more expressive than HyperLTL_S , simple GHyperLTL_{S+C} , and $\text{FO}[\langle, E \rangle]$.
- Simple GHyperLTL_{S+C} is more expressive than simple HyperLTL_S .
- Simple GHyperLTL_{S+C} and HyperLTL_S are expressively incomparable.
- Simple GHyperLTL_{S+C} and $\text{S1S}[E]$ are expressively incomparable.

15:12 Unifying Asynchronous Logics for Hyperproperties

Proof. We first show that there are hyperproperties expressible in simple GHyperLTL_{S+C} but not in HyperLTL_S and in S1S[E] . Given a sentence φ , the *trace property denoted by* φ is the set of traces σ such that the singleton set of traces $\{\sigma\}$ satisfies φ . It is known that HyperLTL_S and S1S[E] capture only *regular* trace properties [7]. In contrast simple GHyperLTL_{S+C} can express powerful non-regular trace properties. For example, consider the so called *suffix property* over $\text{AP} = \{p\}$: a trace σ satisfies the suffix property if there exists a proper suffix σ^k of σ for some $k > 0$ such that $\sigma^k = \sigma$. This non-regular trace property can be expressed in $\text{SHyperLTL}_{S+C}^\emptyset$ as follows:

$$\forall x_1. \forall x_2. \bigwedge_{p \in \text{AP}} \mathbf{G}(p[x_1] \leftrightarrow p[x_2]) \wedge \forall x_1. \exists^P x_2. \left(\bigwedge_{p \in \text{AP}} \mathbf{G}(p[x_1] \leftrightarrow p[x_2]) \wedge \langle x_2 \rangle \mathbf{Y}\top \right)$$

The first conjunct asserts that each model is a singleton, and the second conjunct requires that for the unique trace σ in a model, there is $k > 0$ such that $\sigma(i) = \sigma(i + k)$ for all $i \geq 0$.

Next, we observe that $\text{FO}[\langle, \mathbf{E} \rangle]$ can be easily translated into GHyperLTL_{S+C} , since the pointer quantifiers of GHyperLTL_{S+C} correspond to the quantifiers of $\text{FO}[\langle, \mathbf{E} \rangle]$. Moreover, the predicate $x \leq x'$ of $\text{FO}[\langle, \mathbf{E} \rangle]$, expressing that for the pointed traces (σ, i) and (σ', i') bound to x and x' , $\sigma = \sigma'$ and $i \leq i'$, can be easily captured in GHyperLTL_{S+C} . This is also the case for the equal-level predicate $\mathbf{E}(x, x')$, which can be expressed as $\langle \{x, x'\} \rangle \mathbf{O}(\langle x \rangle \neg \mathbf{Y}\top \wedge \langle x' \rangle \neg \mathbf{Y}\top)$.

In Section 4 we show that model checking of simple GHyperLTL_{S+C} is decidable. Thus, since model checking of both HyperLTL_S and S1S[E] are undecidable [6, 10] and GHyperLTL_{S+C} subsumes HyperLTL_S , by the previous argumentation, the theorem follows. \blacktriangleleft

It remains an open question whether $\text{FO}[\langle, \mathbf{E} \rangle]$ is subsumed by simple GHyperLTL_{S+C} . We conjecture that neither HyperLTL_C nor the fix-point calculus H_μ [20] (which captures both HyperLTL_C and HyperLTL_S [7]) subsume simple GHyperLTL_{S+C} . The motivation for our conjecture is that H_μ sentences consist of a prefix of quantifiers followed by a quantifier-free formula where quantifiers range over *initial* pointed traces $(\sigma, 0)$. Thus, unlike simple GHyperLTL_{S+C} , H_μ cannot express requirements which relate at some point in time an unbounded number of traces. Diagnosability (see Subsection 3.4) falls in this class of requirements. It is known that the following property, which can be easily expressed in simple GHyperLTL_{S+C} , is not definable in HyperLTL [5]: for some $i > 0$, every trace in the given set of traces does not satisfy proposition p at position i . We conjecture that similarly to HyperLTL , such a property (and diagnosability as well) cannot be expressed in H_μ .

Epistemic Temporal Logic KLTL and its relation with GHyperLTL_{S+C} . The logic KLTL [22] is a well-known extension of LTL obtained by adding the unary knowledge modalities \mathbf{K}_a , where a ranges over a finite set Agts of agents, interpreted under the synchronous or asynchronous (perfect recall) semantics. The semantics is given with respect to an observation map $\text{Obs} : \text{Agts} \mapsto 2^{\text{AP}}$ that assigns to each agent a the set of propositions which agent a can observe. Given two finite traces σ and σ' and $a \in \text{Agts}$, σ and σ' are *synchronously equivalent for agent a*, written $\sigma \sim_a^{sy} \sigma'$, if the projections of σ and σ' over $\text{Obs}(a)$ coincide. The finite traces σ and σ' are *asynchronously equivalent for agent a*, written $\sigma \sim_a^{as} \sigma'$, if the projections of $\text{stfr}_{\text{Obs}(a)}(\sigma \cdot P^\omega)$ and $\text{stfr}_{\text{Obs}(a)}(\sigma' \cdot (P')^\omega)$ over $\text{Obs}(a)$ coincide, where P is the last symbol of σ and P' is the last symbol of σ' . For a set of traces \mathcal{L} and a pointed trace (σ, i) over \mathcal{L} , the semantics of the knowledge modalities is as follows, where \sim_a is \sim_a^{sy} under the synchronous semantics, and \sim_a^{as} otherwise: $(\sigma, i) \models_{\mathcal{L}, \text{Obs}} \mathbf{K}_a \varphi \Leftrightarrow$ for all pointed traces (σ', i') on \mathcal{L} such that $\sigma[0, i] \sim_a \sigma'[0, i']$, $(\sigma', i') \models_{\mathcal{L}, \text{Obs}} \varphi$.

We say that \mathcal{L} *satisfies* φ w.r.t. the observation map Obs , written $\mathcal{L} \models_{\text{Obs}} \varphi$, if for all traces $\sigma \in \mathcal{L}$, $(\sigma, 0) \models_{\mathcal{L}, \text{Obs}} \varphi$. The logic KLTL can be easily embedded into GHyperLTL_{S+C} . In particular, the following holds (for details, see Appendix A.2)

► **Theorem 3.3.** *Given an observation map Obs and a KLTL formula ψ over AP , one can construct in linear time a $\text{SHyperLTL}_{S+C}^\emptyset$ sentence φ_\emptyset and a GHyperLTL_{S+C} sentence φ such that φ_\emptyset is equivalent to ψ w.r.t. Obs under the synchronous semantics and φ is equivalent to ψ w.r.t. Obs under the asynchronous semantics. Moreover, φ is a simple GHyperLTL_{S+C} sentence if ψ is in the single-agent fragment of KLTL.*

4 Decidability of Model Checking against Simple GHyperLTL_{S+C}

In this section, we show that (fair) model checking against simple GHyperLTL_{S+C} is decidable. We first prove the result for the fragment $\text{SHyperLTL}_{S+C}^\emptyset$ of simple GHyperLTL_{S+C} by a linear-time reduction to satisfiability of *full* Quantified Propositional Temporal Logic (QPTL, for short) [34], where the latter extends PLTL by quantification over propositions. Then, we show that (fair) model checking of simple GHyperLTL_{S+C} can be reduced in time singly exponential in the size of the formula to fair model checking of $\text{SHyperLTL}_{S+C}^\emptyset$. We also provide optimal complexity bounds for (fair) model checking the fragment $\text{SHyperLTL}_{S+C}^\emptyset$ in terms of a parameter of the given formula called *strong alternation depth*. For this, we first give similar optimal complexity bounds for satisfiability of QPTL.

The syntax of QPTL formulas φ over a finite set AP of atomic propositions is as follows:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathbf{X}\varphi \mid \mathbf{Y}\varphi \mid \varphi \mathbf{U} \varphi \mid \varphi \mathbf{S} \varphi \mid \exists p.\varphi$$

where $p \in AP$ and $\exists p$ is the propositional existential quantifier. A QPTL formula φ is a *sentence* if each proposition p occurs in the scope of a quantifier binding p and each temporal modality occurs in the scope of a quantifier. By introducing \wedge and the operators \mathbf{R} (*release*, dual of \mathbf{U}), \mathbf{P} (*past release*, dual of \mathbf{S}) and $\forall p$ (propositional universal quantifier), every QPTL formula can be converted into an equivalent formula in *negation normal form*, where negation only appears in front of atomic propositions. QPTL formulas are interpreted over pointed traces (σ, i) over AP . All QPTL temporal operators have the same semantics as in PLTL. The semantics of propositional quantification is as follows:

$$(\sigma, i) \models \exists p.\varphi \Leftrightarrow \text{there is a trace } \sigma' \text{ such that } \sigma =_{AP \setminus \{p\}} \sigma' \text{ and } (\sigma', i) \models \varphi$$

where $\sigma =_{AP \setminus \{p\}} \sigma'$ means that the projections of σ and σ' over $AP \setminus \{p\}$ coincide. A formula φ is satisfiable if $(\sigma, 0) \models \varphi$ for some trace σ . We now give a generalization of the standard notion of alternation depth between existential and universal quantifiers which corresponds to the one given in [32] for HyperCTL^* . Our notion takes into account also the occurrences of temporal modalities between quantifier occurrences, but the nesting depth of temporal modalities is not considered (intuitively, it is collapsed to one). Formally, the *strong alternation depth* $sad(\varphi)$ of a QPTL formula φ in negation normal form is inductively defined as follows, where an existential formula is a formula of the form $\exists p.\psi$, a universal formula is of the form $\forall p.\psi$, and for a formula ψ , $\tilde{\psi}$ denotes the negation normal form of $\neg\psi$:

- For $\varphi = p$ and $\varphi = \neg p$ for a given $p \in AP$: $sad(\varphi) := 0$.
- For $\varphi = \varphi_1 \vee \varphi_2$ and for $\varphi = \varphi_1 \wedge \varphi_2$: $sad(\varphi) := \max(\{sad(\varphi_1), sad(\varphi_2)\})$.
- For $\varphi = \exists p.\varphi_1$: if there is no universal sub-formula $\forall\psi$ of φ_1 such that $sad(\forall\psi) = sad(\varphi_1)$, then $sad(\varphi) := sad(\varphi_1)$. Otherwise, $sad(\varphi) := sad(\varphi_1) + 1$.
- For $\varphi = \forall p.\varphi_1$: $sad(\varphi) := sad(\exists p.\tilde{\varphi}_1)$.
- For $\varphi = \mathbf{X}\varphi_1$ or $\varphi = \mathbf{Y}\varphi_1$: $sad(\varphi) := sad(\varphi_1)$.
- For $\varphi = \varphi_1 \mathbf{U} \varphi_2$ or $\varphi = \varphi_1 \mathbf{S} \varphi_2$: let h be the maximum over the strong alternation depths of the universal and existential sub-formulas of φ_1 and φ_2 (the maximum of the empty set is 0). If the following conditions are met, then $sad(\varphi) := h$; otherwise, $sad(\varphi) := h + 1$:

15:14 Unifying Asynchronous Logics for Hyperproperties

- there is no existential or universal sub-formula ψ of φ_1 with $\text{sad}(\psi) = h$;
- there is no universal sub-formula ψ of φ_2 with $\text{sad}(\psi) = h$;
- no existential formula ψ with $\text{sad}(\psi) = h$ occurs in the left operand (resp., right operand) of a sub-formula of φ_2 of the form $\psi_1 \mathcal{O} \psi_2$, where $\mathcal{O} \in \{\mathbf{U}, \mathbf{S}\}$ (resp., $\mathcal{O} \in \{\mathbf{R}, \mathbf{P}\}$).
- Finally, for $\varphi = \varphi_1 \mathbf{R} \varphi_2$ or $\varphi = \varphi_1 \mathbf{P} \varphi_2$: $\text{sad}(\varphi) := \text{sad}(\tilde{\varphi})$.

For example, $\text{sad}(\exists p.(p \mathbf{U} \exists q.q)) = 0$ and $\text{sad}(\exists p.(\exists p.p \mathbf{U} q)) = 1$. The strong alternation depth of an arbitrary QPTL formula corresponds to the one of its negation normal form. The strong alternation depth of a simple $\mathbf{GHyperLTL}_{\mathbf{S}+\mathbf{C}}$ formula is defined similarly but we replace quantification over propositions with quantification over trace variables. For all $n, h \in \mathbb{N}$, $\text{Tower}(h, n)$ denotes a tower of exponentials of height h and argument n : $\text{Tower}(0, n) = n$ and $\text{Tower}(h + 1, n) = 2^{\text{Tower}(h, n)}$. Essentially, the strong alternation depth corresponds to the (unavoidable) power set construction related to the alternation of quantifiers to solve the model-checking problem.

The following result represents an improved version of Theorem 6 in [5] where h -EXPSPACE is the class of languages decided by deterministic Turing machines bounded in space by functions of n in $O(\text{Tower}(h, n^c))$ for some constant $c \geq 1$. While the lower bound directly follows from [34], the upper bound improves the result in [5], since there, occurrences of temporal modalities immediately preceding propositional quantification always count as additional alternations (for details, see Appendix B.1).

► **Theorem 4.1.** *For all $h \geq 0$, satisfiability of QPTL sentences with strong alternation depth at most h is h -EXPSPACE-complete.*

(Fair) Model checking against $\mathbf{SHyperLTL}_{\mathbf{S}+\mathbf{C}}^\emptyset$. We provide now linear-time reductions of (fair) model checking against $\mathbf{SHyperLTL}_{\mathbf{S}+\mathbf{C}}^\emptyset$ to (and from) satisfiability of QPTL which preserve the strong alternation depth. We start with the reduction of (fair) model checking $\mathbf{SHyperLTL}_{\mathbf{S}+\mathbf{C}}^\emptyset$ to QPTL satisfiability.

► **Theorem 4.2.** *Given a fair finite Kripke structure (\mathcal{K}, F) and a $\mathbf{SHyperLTL}_{\mathbf{S}+\mathbf{C}}^\emptyset$ sentence φ , one can construct in linear time a QPTL sentence ψ with the same strong alternation depth as φ such that ψ is satisfiable if and only if $\mathcal{L}(\mathcal{K}, F) \models \varphi$.*

Sketched proof. Let $\mathcal{K} = \langle S, S_0, E, Lab \rangle$. In the reduction of model checking (\mathcal{K}, F) against φ to QPTL satisfiability, we need to merge multiple traces into a unique trace where just one position is considered at any time. An issue is that the hyper quantifiers range over arbitrary pointed traces so that the positions of the different pointed traces in the current trace assignment do not necessarily coincide (intuitively, the different pointed traces are not aligned with respect to the relative current positions). However, we can solve this issue because the offsets between the positions of the pointed traces in the current trace assignment remain constant during the evaluation of the temporal modalities. In particular, assume that (σ, i) is the first pointed trace selected by a hyper quantifier during the evaluation along a path in the syntax tree of φ . We encode σ by keeping track also of the variable x to which (σ, i) is bound and the F -fair path of \mathcal{K} whose associated trace is σ . Let (σ', i') be another pointed trace introduced by another hyper quantifier y during the evaluation of φ . If $i' < i$, we consider an encoding of σ' which is similar to the previous encoding but we precede it with a *padding prefix* of length $i - i'$ of the form $\{\#_{\rightarrow y}\}^{i-i'}$. The arrow \rightarrow indicates that the encoding is along the *forward direction*. Now, assume that $i' > i$. In this case, the encoding of σ' is the merging of two encodings over disjoint sets of propositions: one along the forward direction which encodes the suffix $(\sigma')^{i'-i}$ and another one along the *backward direction* which is of the form $\{\#_{\leftarrow y}\} \cdot \rho \cdot \{\#_{\leftarrow y}\}^\omega$, where ρ is a backward encoding of the *reverse* of

the prefix of σ' of length $i' - i$. In such a way, the encodings of the pointed traces later introduced in the evaluation of φ are aligned with the reference pointed trace (σ, i) . Since the positions in the backward direction overlap some positions in the forward direction, in the translation, we keep track of whether the current position refers to the forward or to the backward direction. The details of the reduction are to QPTL satisfiability can be found in Appendix B.2. \blacktriangleleft

By an adaptation of the known reduction of satisfiability of QPTL without past to model checking of HyperCTL^* [8], we obtain the following result (for details, see Appendix B.3).

► **Theorem 4.3.** *Given a QPTL sentence ψ over AP , one can build in linear time a finite Kripke structure \mathcal{K}_{AP} (depending only on AP) and a singleton-free $\text{SHyperLTL}_{S+C}^\emptyset$ sentence φ having the same strong alternation depth as ψ such that ψ is satisfiable iff $\mathcal{L}(\mathcal{K}_{AP}) \models \varphi$.*

Hence, by Theorems 4.1–4.3, we obtain the following result.

► **Corollary 4.4.** *For all $h \geq 0$, (fair) model checking against $\text{SHyperLTL}_{S+C}^\emptyset$ sentences with strong alternation depth at most h is h -EXPSPACE-complete.*

Reduction to fair model checking against $\text{SHyperLTL}_{S+C}^\emptyset$. We solve the (fair) model checking problem for simple GHyperLTL_{S+C} by a reduction to fair model checking against the fragment $\text{SHyperLTL}_{S+C}^\emptyset$. Our reduction is exponential in the size of the given sentence and is an adaptation of the reduction from model checking simple HyperLTL_S to model checking HyperLTL shown in [6]. As a preliminary step, we first show, by an easy adaptation of the standard automata-theoretic approach for PLTL [37], that the problem for a simple GHyperLTL_{S+C} sentence φ can be reduced in exponential time to the fair model checking problem against a singleton-free sentence in the fragment $\text{SHyperLTL}_{S+C}^\Gamma$ for some set Γ of atomic propositions depending on φ . For details, see Appendix B.4.

► **Proposition 4.5.** *Given a simple GHyperLTL_{S+C} sentence φ and a fair finite Kripke structure (\mathcal{K}, F) over AP , one can build in single exponential time in the size of φ , a fair finite Kripke structure (\mathcal{K}', F') over an extension AP' of AP and a singleton-free $\text{SHyperLTL}_{S+C}^{\Gamma'}$ sentence φ' for some $\Gamma' \subseteq AP'$ such that $\mathcal{L}(\mathcal{K}', F') \models \varphi'$ if and only if $\mathcal{L}(\mathcal{K}, F) \models \varphi$. Moreover, φ' has the same strong alternation depth as φ , $|\varphi'| = O(|\varphi|)$, and $|\mathcal{K}'| = O(|\mathcal{K}| * 2^{O(|\varphi|)})$.*

Let us fix a non-empty finite set $\Gamma \subseteq AP$ of atomic propositions. We now show that fair model checking of the singleton-free fragment of $\text{SHyperLTL}_{S+C}^\Gamma$ can be reduced in polynomial time to fair model checking of $\text{SHyperLTL}_{S+C}^\emptyset$. We observe that in the singleton-free fragment of $\text{SHyperLTL}_{S+C}^\Gamma$, when a pointed trace (σ, i) is selected by a pointed quantifier $\exists^P x$, the positions of σ which are visited during the evaluation of the temporal modalities are all in the (Γ, ω) -stutter factorization of σ with the possible exception of the position i chosen by $\exists^P x$. Thus, given a set \mathcal{L} of traces and a special proposition $\# \notin AP$, we define an extension $\text{stfr}_\Gamma^\#(\mathcal{L})$ of the set $\text{stfr}_\Gamma(\mathcal{L}) = \{\text{stfr}_\Gamma(\sigma) \mid \sigma \in \mathcal{L}\}$ as follows. Intuitively, we consider for each trace $\sigma \in \mathcal{L}$, its Γ -stutter trace $\text{stfr}_\Gamma(\sigma)$ and the extensions of $\text{stfr}_\Gamma(\sigma)$ which are obtained by adding an extra position marked by proposition $\#$ (this extra position does not belong to the (Γ, ω) -stutter factorization of σ). Formally, $\text{stfr}_\Gamma^\#(\mathcal{L})$ is the smallest set containing $\text{stfr}_\Gamma(\mathcal{L})$ and satisfying the following condition:

- for each trace $\sigma \in \mathcal{L}$ with (Γ, ω) -stutter factorization $\{\ell_k\}_{k \geq 0}$ and position $i \in (\ell_k, \ell_{k+1})$ for some $k \geq 0$, the trace $\sigma(\ell_0) \dots \sigma(\ell_k) (\sigma(i) \cup \{\#\}) \sigma(\ell_{k+1}) \sigma(\ell_{k+2}) \dots \in \text{stfr}_\Gamma^\#(\mathcal{L})$.

Given a singleton-free formula φ in $\text{SHyperLTL}_{S+C}^\Gamma$, we denote by $\text{T}_\#(\varphi)$ the formula in $\text{SHyperLTL}_{S+C}^\emptyset$ obtained from φ by applying inductively the following transformations:

15:16 Unifying Asynchronous Logics for Hyperproperties

- the Γ -relativized temporal modalities are replaced with their \emptyset -relativized counterparts;
- each formula $\exists^P x. \phi$ is replaced with $\exists^P x. (\mathsf{T}_{\#}(\phi) \wedge \langle x \rangle (\mathbf{XG} \neg \#[x] \wedge (\mathbf{YT} \rightarrow \mathbf{YH} \neg \#[x])))$.

Intuitively, the formula $\mathsf{T}_{\#}(\exists^P x. \phi)$ states that for the pointed trace (σ, i) selected by the pointed quantifier, at most position i may be marked by the special proposition $\#$. By the semantics of the logics considered, the following holds.

► **Remark 4.6.** Given a singleton-free sentence φ of $\text{SHyperLTL}_{S+C}^{\Gamma}$ and a set \mathcal{L} of traces, it holds that $\mathcal{L} \models \varphi$ if and only if $\text{stfr}_{\Gamma}^{\#}(\mathcal{L}) \models \mathsf{T}_{\#}(\varphi)$.

Let us fix now a fair finite Kripke structure (\mathcal{K}, F) . We first show that one can build in polynomial time a finite Kripke structure $(\mathcal{K}_{\Gamma}, F_{\Gamma})$ and a LTL formula θ_{Γ} such that $\text{stfr}_{\Gamma}^{\#}(\mathcal{L}(\mathcal{K}, F))$ coincides with the traces of $\mathcal{L}(\mathcal{K}_{\Gamma}, F_{\Gamma})$ satisfying θ_{Γ} (details are in Appendix B.5).

► **Proposition 4.7.** *Given $\emptyset \neq \Gamma \subseteq AP$ and a fair finite Kripke structure (\mathcal{K}, F) over AP , one can construct in polynomial time a fair finite Kripke structure $(\mathcal{K}_{\Gamma}, F_{\Gamma})$ and a LTL formula θ_{Γ} such that $\text{stfr}_{\Gamma}^{\#}(\mathcal{L}(\mathcal{K}, F))$ is the set of traces $\sigma \in \mathcal{L}(\mathcal{K}_{\Gamma}, F_{\Gamma})$ so that $\sigma \models \theta_{\Gamma}$.*

Fix now a singleton-free sentence φ of $\text{SHyperLTL}_{S+C}^{\Gamma}$. For the given fair finite Kripke structure (\mathcal{K}, F) over AP , let $(\mathcal{K}_{\Gamma}, F_{\Gamma})$ and θ_{Γ} as in the statement of Proposition 4.7. We consider the $\text{SHyperLTL}_{S+C}^{\emptyset}$ sentence $\mathsf{T}(\varphi)$ obtained from $\mathsf{T}_{\#}(\varphi)$ by inductively replacing each subformula $\exists^P x. \phi$ of $\mathsf{T}_{\#}(\varphi)$ with $\exists^P x. (\mathsf{T}(\phi) \wedge \langle x \rangle \mathbf{O}(\neg \mathbf{YT} \wedge \theta_{\Gamma}[x]))$. In other terms, we ensure that in $\mathsf{T}_{\#}(\varphi)$ the hyper quantification ranges over traces which satisfy the LTL formula θ_{Γ} . By Remark 4.6 and Proposition 4.7, we obtain that $\mathcal{L}(\mathcal{K}, F) \models \varphi$ if and only if $\mathcal{L}(\mathcal{K}_{\Gamma}, F_{\Gamma}) \models \mathsf{T}(\varphi)$. Thus, together with Proposition 4.5, we obtain the following result.

► **Theorem 4.8.** *The (fair) model checking problem against simple GHyperLTL_{S+C} can be reduced in singly exponential time to fair model checking against $\text{SHyperLTL}_{S+C}^{\emptyset}$.*

5 Conclusion

We have introduced a novel hyper logic GHyperLTL_{S+C} which merges two known asynchronous temporal logics for hyperproperties, namely stuttering HyperLTL and context HyperLTL . Even though model checking of the resulting logic GHyperLTL_{S+C} is undecidable, we have identified a useful fragment, called *simple* GHyperLTL_{S+C} , that has a decidable model checking, is strictly more expressive than HyperLTL and than previously proposed fragments of asynchronous temporal logics for hyperproperties with a decidable model checking. For the fragment $\text{SHyperLTL}_{S+C}^{\emptyset}$ of *simple* GHyperLTL_{S+C} , we have given optimal complexity bounds of (fair) model checking in terms of the strong alternation depth of the given sentence. For arbitrary sentences in *simple* GHyperLTL_{S+C} , (fair) model checking is reduced in exponential time to fair model checking of $\text{SHyperLTL}_{S+C}^{\emptyset}$. It is worth noting that *simple* GHyperLTL_{S+C} can express non-regular trace properties over singleton sets of traces which are not definable in S1S[E] . An intriguing open question is whether $\text{FO}[<,E]$ can be embedded in *simple* GHyperLTL_{S+C} . In a companion paper, we study asynchronous properties on finite traces by adapting *simple* GHyperLTL_{S+C} in prenex form to finite traces, and introduce practical model-checking algorithms for useful fragments of this logic.

References

- 1 Jan Baumeister, Norine Coenen, Borzoo Bonakdarpour, Bernd Finkbeiner, and César Sánchez. A Temporal Logic for Asynchronous Hyperproperties. In *Proc. 33rd CAV*, volume 12759 of *LNCS 12759*, pages 694–717. Springer, 2021. doi:10.1007/978-3-030-81685-8_33.

- 2 Raven Beutner, Bernd Finkbeiner, Hadar Frenkel, and Niklas Metzger. Second-Order Hyperproperties. In *Proc. 35th CAV*, volume 13965 of *Lecture Notes in Computer Science*, pages 309–332. Springer, 2023. doi:10.1007/978-3-031-37703-7_15.
- 3 Benjamin Bittner, Marco Bozzano, Alessandro Cimatti, Marco Gario, Stefano Tonetta, and Viktoria Vozárová. Diagnosability of fair transition systems. *Artif. Intell.*, 309:103725, 2022. doi:10.1016/J.ARTINT.2022.103725.
- 4 Marco Bozzano, Alessandro Cimatti, Marco Gario, and Stefano Tonetta. Formal Design of Asynchronous Fault Detection and Identification Components using Temporal Epistemic Logic. *Log. Methods Comput. Sci.*, 11(4), 2015. doi:10.2168/LMCS-11(4:4)2015.
- 5 Laura Bozzelli, Bastien Maubert, and Spophie Pinchinat. Unifying Hyper and Epistemic Temporal Logics. In *Proc. 18th FoSSaCS*, LNCS 9034, pages 167–182. Springer, 2015. doi:10.1007/978-3-662-46678-0_11.
- 6 Laura Bozzelli, Adriano Peron, and César Sánchez. Asynchronous Extensions of HyperLTL. In *Proc. 36th LICS*, pages 1–13. IEEE, 2021. doi:10.1109/LICS52264.2021.9470583.
- 7 Laura Bozzelli, Adriano Peron, and César Sánchez. Expressiveness and Decidability of Temporal Logics for Asynchronous Hyperproperties. In *Proc. 33rd CONCUR*, volume 243 of *LIPICs*, pages 27:1–27:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPICs.CONCUR.2022.27.
- 8 Michael R. Clarkson, Bernd Finkbeiner, Masoud Koleini, Kristopher K. Micinski, Markus N. Rabe, and César Sánchez. Temporal Logics for Hyperproperties. In *Proc. 3rd POST*, LNCS 8414, pages 265–284. Springer, 2014. doi:10.1007/978-3-642-54792-8_15.
- 9 Michael R. Clarkson and Fred B. Schneider. Hyperproperties. *Journal of Computer Security*, 18(6):1157–1210, 2010. doi:10.3233/JCS-2009-0393.
- 10 Norine Coenen, Bernd Finkbeiner, Christopher Hahn, and Jana Hofmann. The hierarchy of hyperlogics. In *Proc. 34th LICS*, pages 1–13. IEEE, 2019. doi:10.1109/LICS.2019.8785713.
- 11 Rayna Dimitrova, Bernd Finkbeiner, Máté M Kovács, Markus N. Rabe, and Helmut Seidl. Model Checking Information Flow in Reactive Systems. In *Proc. 13th VMCAI*, LNCS 7148, pages 169–185. Springer, 2012. doi:10.1007/978-3-642-27940-9_12.
- 12 E. Allen Emerson and Joseph Y. Halpern. “Sometimes” and “Not Never” revisited: on branching versus linear time temporal logic. *J. ACM*, 33(1):151–178, 1986. doi:10.1145/4904.4999.
- 13 Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning about knowledge*, volume 4. MIT Press Cambridge, 1995. doi:10.7551/mitpress/5803.001.0001.
- 14 Bernd Finkbeiner and Christopher Hahn. Deciding Hyperproperties. In *Proc. 27th CONCUR*, *LIPICs* 59, pages 13:1–13:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPICs.CONCUR.2016.13.
- 15 Bernd Finkbeiner and Martin Zimmermann. The first-order logic of hyperproperties. In *Proc. 34th STACS*, *LIPICs* 66, pages 30:1–30:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017. doi:10.4230/LIPICs.STACS.2017.30.
- 16 Michael J. Fischer and Richard E. Ladner. Propositional Dynamic Logic of Regular Programs. *J. Comput. Syst. Sci.*, 18(2):194–211, 1979. doi:10.1016/0022-0000(79)90046-1.
- 17 Joseph A. Goguen and José Meseguer. Security Policies and Security Models. In *IEEE Symposium on Security and Privacy*, pages 11–20. IEEE Computer Society, 1982. doi:10.1109/SP.1982.10014.
- 18 Jens Oliver Gutsfeld, Arne Meier, Christoph Ohrem, and Jonni Virtema. Temporal Team Semantics Revisited. In *Proc. 37th LICS*, pages 44:1–44:13. ACM, 2022. doi:10.1145/3531130.3533360.
- 19 Jens Oliver Gutsfeld, Markus Müller-Olm, and Christoph Ohrem. Propositional dynamic logic for hyperproperties. In *Proc. 31st CONCUR*, *LIPICs* 171, pages 50:1–50:22. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPICs.CONCUR.2020.50.

- 20 Jens Oliver Gutsfeld, Markus Müller-Olm, and Christoph Ohrem. Automata and fixpoints for asynchronous hyperproperties. *Proc. ACM Program. Lang.*, 4(POPL), 2021. doi:10.1145/3434319.
- 21 Joseph Y. Halpern and Kevin R. O’Neill. Secrecy in multiagent systems. *ACM Trans. Inf. Syst. Secur.*, 12(1), 2008.
- 22 Joseph Y. Halpern and Moshe Y. Vardi. The Complexity of Reasoning about Knowledge and Time: Extended Abstract. In *Proc. 18th STOC*, pages 304–315. ACM, 1986. doi:10.1145/12130.12161.
- 23 Andreas Krebs, Arne Meier, Jonni Virtema, and Martin Zimmermann. Team Semantics for the Specification and Verification of Hyperproperties. In *Proc. 43rd MFCS*, LIPIcs 117, pages 10:1–10:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPIcs.MFCS.2018.10.
- 24 Orna Kupferman, Nir Piterman, and Moshe Y. Vardi. From liveness to promptness. *Formal Methods Syst. Des.*, 34(2):83–103, 2009. doi:10.1007/S10703-009-0067-Z.
- 25 Orna Kupferman and Moshe Y. Vardi. Weak alternating automata are not that weak. *ACM Transactions on Computational Logic*, 2(3):408–429, 2001. doi:10.1145/377978.377993.
- 26 Orna Kupferman, Moshe Y. Vardi, and Pierre Wolper. An Automata-Theoretic Approach to Branching-Time Model Checking. *J. ACM*, 47(2):312–360, 2000. doi:10.1145/333979.333987.
- 27 Martin Lück. On the complexity of linear temporal logic with team semantics. *Theor. Comput. Sci.*, 837:1–25, 2020. doi:10.1016/j.tcs.2020.04.019.
- 28 Zohar Manna and Amir Pnueli. *The Temporal Logic of Reactive and Concurrent Systems - Specification*. Springer-Verlag, 1992. doi:10.1007/978-1-4612-0931-7.
- 29 John D. McLean. A General Theory of Composition for a Class of "Possibilistic" Properties. *IEEE Trans. Software Eng.*, 22(1):53–67, 1996. doi:10.1109/32.481534.
- 30 S. Miyano and T. Hayashi. Alternating finite automata on ω -words. *Theoretical Computer Science*, 32:321–330, 1984. doi:10.1016/0304-3975(84)90049-5.
- 31 Amir Pnueli. The Temporal Logic of Programs. In *Proc. 18th FOCS*, pages 46–57. IEEE Computer Society, 1977. doi:10.1109/SFCS.1977.32.
- 32 Markus N. Rabe. *A temporal logic approach to information-flow control*. PhD thesis, Saarland University, 2016.
- 33 Meera Sampath, Raja Sengupta, Stephen Lafortune, Kazin Sinnamohideen, and Demosthenis Teneketzis. Diagnosability of discrete-event systems. *IEEE Trans. Autom. Control.*, 40(9):1555–1575, 1995. doi:10.1109/9.412626.
- 34 A. Prasad Sistla, Moshe Y. Vardi, and Pierre Wolper. The Complementation Problem for Büchi Automata with Applications to Temporal Logic. *Theoretical Computer Science*, 49:217–237, 1987. doi:10.1016/0304-3975(87)90008-9.
- 35 Ron van der Meyden and Nikolay V. Shilov. Model checking knowledge and time in systems with perfect recall (extended abstract). In *Proc. 19th FSTTCS*, LNCS 1738, pages 432–445. Springer, 1999. doi:10.1007/3-540-46691-6_35.
- 36 Moshe Y. Vardi. A temporal fixpoint calculus. In *Proc. 15th POPL*, pages 250–259. ACM, 1988.
- 37 Moshe Y. Vardi and Pierre Wolper. Reasoning about infinite computations. *Inf. Comput.*, 115(1):1–37, 1994. doi:10.1006/inco.1994.1092.
- 38 Jonni Virtema, Jana Hofmann, Bernd Finkbeiner, Juha Kontinen, and Fan Yang. Linear-Time Temporal Logic with Team Semantics: Expressivity and Complexity. In *Proc. 41st IARCS FSTTCS*, LIPIcs 213, pages 52:1–52:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.FSTTCS.2021.52.
- 39 Steve Zdancewic and Andrew C. Myers. Observational Determinism for Concurrent Program Security. In *Proc. 16th IEEE CSFW-16*, pages 29–43. IEEE Computer Society, 2003. doi:10.1109/CSFW.2003.1212703.

- 40 W. Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theoretical Computer Science*, 200(1-2):135–183, 1998. doi:10.1016/S0304-3975(98)00009-7.

A Proofs from Section 3

A.1 Equally-expressive extension of simple GHyperLTL_{S+C}

In this section, we show that the extension of simple GHyperLTL_{S+C} as defined at the end of Subsection 3.3 is not more expressive than simple GHyperLTL_{S+C}. In the following, a *singleton-context* formula is a quantifier-free GHyperLTL_{S+C} formula of the form $\langle x \rangle \xi$, where ξ only uses singleton contexts $\langle y \rangle$ and temporal modalities with subscript \emptyset . A singleton-context formula is *simple* if it is of the form $\langle x \rangle \psi[x]$ for some PLTL formula ψ . Thus, the considered extension of simple GHyperLTL_{S+C} is obtained by replacing simple singleton-context sub-formulas with arbitrary singleton-context sub-formulas. Given two quantifier-free GHyperLTL_{S+C} formulas φ and φ' , we say that φ and φ' are *equivalent* if (i) φ and φ' use the same trace variables, and (ii) for each trace assignment Π whose domain contains the variables of φ , $(\Pi, \text{VAR}) \models \varphi$ iff $(\Pi, \text{VAR}) \models \varphi'$. In order to show that the considered extension of simple GHyperLTL_{S+C} has the same expressiveness as simple GHyperLTL_{S+C}, it suffices to show the following result.

► **Proposition A.1.** *Given a singleton-context formula $\langle x \rangle \xi$, one can construct a Boolean combination φ of simple singleton-context formulas and relativized propositions such that φ and $\langle x \rangle \xi$ are equivalent.*

Proof. For the proof, we need additional definitions. Let Λ be a finite set whose elements are simple singleton-context formulas or relativized atomic propositions. A *guess* for Λ is a mapping $H : \Lambda \mapsto \{\top, \neg\top\}$ assigning to each formula in Λ a Boolean value (\top for *true*, and $\neg\top$ for *false*). We denote by G_Λ the finite set of guesses for Λ .

Fix a singleton-context formula $\langle x \rangle \xi$ which is not simple. We prove Proposition A.1 by induction on the nesting depth of the context modalities in ξ .

For the base case, ξ does not contain context modalities. Let Λ be the set of relativized atomic propositions $p[y]$ occurring in ξ such that $y \neq x$. Then, the formula φ satisfying Proposition A.1 is given by

$$\varphi := \bigvee_{H \in G_\Lambda} (\xi(H) \wedge \bigwedge_{\theta \in \{\theta \in \Lambda \mid H(\theta) = \top\}} \theta \wedge \bigwedge_{\theta \in \{\theta \in \Lambda \mid H(\theta) = \neg\top\}} \neg\theta)$$

where $\xi(H)$ is obtained from ξ by replacing all occurrences of the sub-formulas $\theta \in \Lambda$ in ξ with $H(\theta)$. Note that φ is a Boolean combination of simple singleton-context formulas and relativized propositions. Correctness of the construction follows from the fact that by the semantics of GHyperLTL_{S+C}, the position of the pointed trace assigned to a variable y distinct from x remains unchanged during the valuation of the formula ξ within the singleton context $\langle x \rangle$.

For the induction step, assume that ξ contains singleton context modalities. By the induction hypothesis, one can construct a formula ξ' which is a Boolean combination of simple singleton-context formulas and relativized propositions such that $\langle x \rangle \xi$ and $\langle x \rangle \xi'$ are equivalent. Let Λ be the set consisting of the relativized atomic propositions $p[y]$ occurring in ξ' with $y \neq x$ which are not in scope of a context modality, and the sub-formulas of ξ' of the form $\langle y \rangle \theta$ such that $y \neq x$ (note that by hypothesis $\langle y \rangle \theta$ is a simple singleton-context formula). Then, the formula φ satisfying Proposition A.1 is given by

$$\varphi := \bigvee_{H \in G_\Lambda} (\xi'(H) \wedge \bigwedge_{\theta \in \{\theta \in \Lambda \mid H(\theta) = \top\}} \theta \wedge \bigwedge_{\theta \in \{\theta \in \Lambda \mid H(\theta) = \neg\top\}} \neg\theta)$$

where $\xi'(H)$ is obtained from ξ' by replacing all occurrences of the sub-formulas $\theta \in \Lambda$ in ξ' with $H(\theta)$, and by removing the singleton context modality $\langle x \rangle$. Correctness of the construction directly follows from the induction hypothesis and the semantics of GHyperLTL_{S+C} . ◀

A.2 Proof of Theorem 3.3

► **Theorem 3.3.** *Given an observation map Obs and a KLTL formula ψ over AP , one can construct in linear time a $\text{SHyperLTL}_{S+C}^\emptyset$ sentence φ_\emptyset and a GHyperLTL_{S+C} sentence φ such that φ_\emptyset is equivalent to ψ w.r.t. Obs under the synchronous semantics and φ is equivalent to ψ w.r.t. Obs under the asynchronous semantics. Moreover, φ is a simple GHyperLTL_{S+C} sentence if ψ is in the single-agent fragment of KLTL.*

Proof. We focus on the construction of the GHyperLTL_{S+C} sentence φ capturing the KLTL formula ψ under the asynchronous semantics w.r.t. the given observation map Obs . The construction of the $\text{SHyperLTL}_{S+C}^\emptyset$ sentence φ_\emptyset capturing ψ under the synchronous semantics w.r.t. Obs is similar. We inductively define a mapping T_{Obs} assigning to each pair (ϕ, x) consisting of a KLTL formula ϕ and a trace variable $x \in \text{VAR}$, a GHyperLTL_{S+C} formula $T_{\text{Obs}}(\phi, x)$. Intuitively, x is associated to the current evaluated pointed trace. The mapping T_{Obs} is homomorphic w.r.t. the Boolean connectives and the LTL temporal modalities (i.e., $T(\mathcal{O}\phi, x) := \mathcal{O}T(\phi, x)$ for each $\mathcal{O} \in \{\neg, \mathbf{X}\}$ and $T(\phi_1 \mathcal{O} \phi_2, x) := T(\phi_1, x) \mathcal{O} T(\phi_2, x)$ for each $\mathcal{O} \in \{\vee, \mathbf{U}\}$) and is defined as follows when the given KLTL formula is an atomic proposition or its root operator is a knowledge modality:

- $T(p, x) := p[x]$;
- $T(\mathbf{K}_a\phi, x) := \forall^P y. (\theta_{\text{Obs}}(a, x, y) \rightarrow T_{\text{Obs}}(\phi, y))$ where $y \neq x$ and
 $\theta_{\text{Obs}}(a, x, y) := \bigwedge_{p \in \text{Obs}(a)} \mathbf{H}_{\text{Obs}(a)}(p[x] \leftrightarrow p[y]) \wedge \mathbf{O}_{\text{Obs}(a)}(\langle x \rangle \neg \mathbf{Y} \top \wedge \langle y \rangle \neg \mathbf{Y} \top)$

Intuitively, $\theta_{\text{Obs}}(a, x, y)$ asserts that for the pointed traces (σ, i) and (σ', i') bound to the trace variables x and y , respectively, $\sigma[0, i]$ and $\sigma'[0, i']$ are asynchronously equivalent for agent a w.r.t. the given observation map Obs . Note that $T_{\text{Obs}}(\phi, x)$ is a simple GHyperLTL_{S+C} formula if ϕ is in the one-agent fragment of KLTL. Let \mathcal{L} be a set of traces, $x \in \text{VAR}$, (σ, i) be a pointed trace over \mathcal{L} , and Π be a trace assignment over \mathcal{L} such that $x \in \text{Dom}(\Pi)$ and $\Pi(x) = (\sigma, i)$. By a straightforward induction on the structure of a KLTL formula φ , one can show that $(\sigma, i) \models_{\mathcal{L}, \text{Obs}} \phi$ if and only if $(\Pi, \text{VAR}) \models_{\mathcal{L}} T_{\text{Obs}}(\phi, x)$. Hence, the desired GHyperLTL_{S+C} sentence φ is given by $\forall x. T_{\text{Obs}}(\psi, x)$ and we are done. ◀

B Proofs from Section 4

B.1 Automata for QPTL and Upper bounds of Theorem 4.1

B.1.1 Automata for QPTL

In this section, we recall the classes of automata exploited in [5] for solving satisfiability of full QPTL with past. In the standard automata-theoretic approach for QPTL formulas φ in prenex form [34], first, one converts the quantifier-free part ψ of φ into an equivalent nondeterministic Büchi automaton (NBA) accepting the set $\mathcal{L}(\psi)$ of traces satisfying ψ . Then, by using the closure of NBA definable languages under projection and complementation, one obtains an NBA accepting $\mathcal{L}(\varphi)$. This approach would not work for arbitrary QPTL formulas φ , where quantifiers can occur in the scope of both past and future temporal modalities. In this case, for a subformula φ' of φ , we need to keep track of the full set $\mathcal{L}_\varphi(\varphi')$ of pointed traces satisfying φ , and not simply $\mathcal{L}(\varphi')$. Thus, the automata-theoretic approach proposed in [5] is based on a compositional translation of QPTL formulas into a

simple two-way extension of NBA, called *simple two-way* NBA (SNBA, for short) accepting languages of *pointed traces*. Moreover, each step of the translation into SNBA is based on an intermediate formalism consisting in a two-way extension of the class of (one-way) *hesitant alternating automata* (HAA, for short) over infinite words introduced in [26]. We now recall these two classes of automata. Moreover, for the class of two-way HAA, we consider additional requirements which may be fulfilled by some states of the automaton. These extra requirements allow to obtain a more fine-grained complexity analysis in the translation of two-way HAA into equivalent SNBA, and they are important for obtaining an asymptotically optimal automata-theoretic approach for QPTL in terms of the strong alternation depth of a QPTL formula.

SNBA [5]. An SNBA over traces on AP is a tuple $\mathcal{A} = \langle Q, Q_0, \delta, F_-, F_+ \rangle$, where Q is a finite set of states, $Q_0 \subseteq Q$ is a set of initial states, $\delta : Q \times \{\rightarrow, \leftarrow\} \times 2^{\text{AP}} \rightarrow 2^Q$ is a transition function, and F_- and F_+ are sets of accepting states. Intuitively, the symbols \rightarrow and \leftarrow are used to denote forward and backward moves. A run of \mathcal{A} over a pointed trace (σ, i) is a pair $r = (r_{\leftarrow}, r_{\rightarrow})$ such that $r_{\rightarrow} = q_i, q_{i+1} \dots$ is an infinite sequence of states, $r_{\leftarrow} = q'_i, q'_{i-1} \dots q'_0 q'_{-1}$ is a finite sequence of states, and: (i) $q_i = q'_i \in Q_0$; (ii) for each $h \geq i$, $q_{h+1} \in \delta(q_h, \rightarrow, \sigma(h))$; and (iii) for each $h \in [0, i]$, $q'_{h-1} \in \delta(q'_h, \leftarrow, \sigma(h))$.

Thus, starting from the initial position i in the input pointed trace (σ, i) , the automaton splits in two copies: the first one moves forwardly along the suffix of σ starting from position i and the second one moves backwardly along the prefix $\sigma(0) \dots \sigma(i)$. The run $r = (r_{\leftarrow}, r_{\rightarrow})$ is *accepting* if $q'_{-1} \in F_-$ and r_{\rightarrow} visits infinitely often some state in F_+ . A pointed trace (σ, i) is accepted by \mathcal{A} if there is an accepting run of \mathcal{A} over (σ, i) . We denote by $\mathcal{L}_\varphi(\mathcal{A})$ the set of pointed traces accepted by \mathcal{A} and by $\mathcal{L}(\mathcal{A})$ the set of traces σ such that $(\sigma, 0) \in \mathcal{L}_\varphi(\mathcal{A})$.

Two-way HAA [5]. Now, we recall the class of two-way HAA. For a set X , $\mathcal{B}^+(X)$ denotes the set of positive Boolean formulas over X built from elements in X using \vee and \wedge (we also allow the formulas **true** and **false**). For a formula $\theta \in \mathcal{B}^+(X)$, a *model* Y of θ is a subset Y of X which satisfies θ . The model Y of θ is *minimal* if no strict subset of Y satisfies θ .

A two-way HAA \mathcal{A} over traces on AP is a tuple $\mathcal{A} = \langle Q, q_0, \delta, F_-, F \rangle$, where Q is a finite set of states, $q_0 \in Q$ is the initial state, $\delta : Q \times 2^{\text{AP}} \rightarrow \mathcal{B}^+(\{\rightarrow, \leftarrow\} \times Q)$ is a transition function, $F_- \subseteq Q$ is the *backward acceptance condition*, and F is a *strata family* encoding a particular kind of parity acceptance condition and imposing some syntactical constraints on the transition function δ . Before defining F , we give the notion of run which is independent of F and F_- . We restrict ourselves to *memoryless* runs, in which the behavior of the automaton depends only on the current input position and current state. Since later we will deal only with parity acceptance conditions, memoryless runs are sufficient (see e.g. [40]). Formally, given a pointed trace (σ, i) and a state $q' \in Q$, an (i, q') -run of \mathcal{A} over σ is a directed graph $\langle V, E, v_0 \rangle$ with set of vertices $V \subseteq (\mathbb{N} \cup \{-1\}) \times Q$ and initial vertex $v_0 = (i, q')$. Intuitively, a vertex (j, q) with $j \geq 0$ describes a copy of the automaton which is in state q and reads the j^{th} input position. Additionally, we require that the set of edges E is consistent with the transition function δ . Formally, for every vertex $v = (j, q) \in V$ such that $j \geq 0$, there is a *minimal* model $\{(dir_1, q_1), \dots, (dir_n, q_n)\}$ of $\delta(q, \sigma(j))$ such that the set of successors of $v = (j, q)$ is $\{(j_1, q_1), \dots, (j_n, q_n)\}$, where for all $k \in [1, n]$, $j_k = j + 1$ if $dir_k = \rightarrow$, and $j_k = j - 1$ otherwise.

An infinite path π of a run is *eventually strictly-forward* whenever π has a suffix of the form $(i, q_1), (i + 1, q_2), \dots$ for some $i \geq 0$.

Now, we formally define F and give the semantic notion of acceptance. F is a *strata family* of the form $F = \{\langle \rho_1, Q_1, F_1 \rangle, \dots, \langle \rho_k, Q_k, F_k \rangle\}$, where Q_1, \dots, Q_k is a partition of the set of

states Q of \mathcal{A} , and for all $i \in [1, k]$, $\rho_i \in \{-, \mathfrak{t}, \mathbf{B}, \mathbf{C}\}$ and $F_i \subseteq Q_i$, such that $F_i = \emptyset$ whenever $\rho_i \in \{\mathfrak{t}, -\}$. A stratum $\langle \rho_i, Q_i, F_i \rangle$ is called a *negative* stratum if $\rho_i = -$, a *transient* stratum if $\rho_i = \mathfrak{t}$, a Büchi stratum (with Büchi acceptance condition F_i) if $\rho_i = \mathbf{B}$, and a coBüchi stratum (with coBüchi acceptance condition F_i) if $\rho_i = \mathbf{C}$. Additionally, there is a partial order \leq on the sets Q_1, \dots, Q_k such that the following holds:

- *Partial order requirement.* Moves from states in Q_i lead to states in components Q_j such that $Q_j \leq Q_i$; additionally, if Q_i belongs to a transient stratum, there are no moves from Q_i leading to Q_i .
- *Eventually syntactical requirement.* For all moves (dir, q') from states $q \in Q_i$ such that $q' \in Q_i$ as well, the following holds: dir is \leftarrow if the stratum of Q_i is negative, and dir is \rightarrow otherwise.
- Each component Q_i satisfies one of the following two requirements:
 - *Existential requirement:* for all states $q_i \in Q_i$ and input symbols a , if $\delta(q, a)$ is rewritten in disjunctive normal form, then each conjunction contains at most one occurrence of a state in Q_i .
 - *Universal requirement:* for all states $q_i \in Q_i$ and input symbols a , if $\delta(q, a)$ is rewritten in conjunctive normal form, then each disjunction contains at most one occurrence of a state in Q_i .
- Components of Büchi strata satisfy the existential requirement, while components of coBüchi strata satisfy the universal requirement.

The partial order requirement ensures that every infinite path π of a run gets trapped in the component Q_i of some stratum. The *eventually syntactical requirement* ensures that Q_i belongs to a Büchi or coBüchi stratum and that π is eventually strictly-forward. Moreover, the existential requirement for a component Q_i establishes that from each state $q \in Q_i$, at most one copy of the automaton is sent to the next input symbol in component Q_i (all the other copies move to states in strata with order lower than Q_i). Finally, the universal requirement corresponds to the dual of the existential requirement. Note that transient components trivially satisfy both the existential and the universal requirement. It is worth noting that the existential and universal requirements are not considered in [5]. On the other hand, they are crucial in the automata-theoretic approach for QPTL.

Now we define when a run is accepting. Let π be an infinite path of a run, $\langle \rho_i, Q_i, F_i \rangle$ be the Büchi or coBüchi stratum in which π gets trapped, and $Inf(\pi)$ be the states from Q that occur infinitely many times in π . The path π is *accepting* whenever $Inf(\pi) \cap F_i \neq \emptyset$ if $\rho_i = \mathbf{B}$ and $Inf(\pi) \cap F_i = \emptyset$ otherwise (i.e. π satisfies the corresponding Büchi or coBüchi requirement). A run is *accepting* if: (i) all its infinite paths are accepting and (ii) for each vertex $(-1, q)$ reachable from the initial vertex, it holds that $q \in F_-$ (recall that F_- is the backward acceptance condition of \mathcal{A}). The ω -pointed language $\mathcal{L}_p(\mathcal{A})$ of \mathcal{A} is the set of pointed traces (σ, i) such that there is an accepting (i, q_0) -run of \mathcal{A} on σ .

The *dual automaton* $\tilde{\mathcal{A}}$ of a two-way HAA $\mathcal{A} = \langle Q, q_0, \delta, F_-, F \rangle$ is defined as $\tilde{\mathcal{A}} = \langle Q, q_0, \tilde{\delta}, Q \setminus F_-, \tilde{F} \rangle$, where $\tilde{\delta}(q, a)$ is the dual formula of $\delta(q, a)$ (obtained from $\delta(q, a)$ by switching \vee and \wedge , and switching **true** and **false**), and \tilde{F} is obtained from F by converting a Büchi stratum $\langle \mathbf{B}, Q_i, F_i \rangle$ into the coBüchi stratum $\langle \mathbf{C}, Q_i, F_i \rangle$ and a coBüchi stratum $\langle \mathbf{C}, Q_i, F_i \rangle$ into the Büchi stratum $\langle \mathbf{B}, Q_i, F_i \rangle$. By construction the dual automaton $\tilde{\mathcal{A}}$ of \mathcal{A} is still a two-way HAA. Following standard arguments (see e.g. [40]), the following holds.

► **Proposition B.1** ([5]). *The dual automaton $\tilde{\mathcal{A}}$ of a two-way HAA \mathcal{A} is a two-way HAA accepting the complement of $\mathcal{L}_p(\mathcal{A})$.*

Note that SNBA correspond to a subclass of two-way HAA.

► **Proposition B.2** ([5]). *An SNBA \mathcal{A} can be converted “on the fly” in linear time into a two-way HAA accepting $\mathcal{L}_\varphi(\mathcal{A})$.*

Additional requirements on components of two-way HAA. Let $\mathcal{A} = \langle Q, q_0, \delta, F_-, F \rangle$ be a two-way HAA. A component Q_i of \mathcal{A} is *globally nondeterministic* if the following inductively holds:

- Q_i satisfies the existential requirement and Q_i is not a coBüchi component;
- for each component Q_k distinct from Q_i such that there are moves from Q_k leading to Q_i (hence, $Q_k > Q_i$), (i) Q_k is *globally nondeterministic*, and (ii) for each $q \in Q_k$ and input symbol a , if $\delta(q, a)$ is rewritten in disjunctive normal form, then each conjunction contains at most one occurrence of a state in Q_i .

The previous requirement ensures that in each run r and for each input position ℓ , there is at most one copy of the automaton that is in some state of Q_i reading position ℓ . A state of \mathcal{A} is *globally nondeterministic* if it belongs to some globally nondeterministic component of \mathcal{A} .

From two-way HAA to SNBA [5]. Two-way HAA can be translated in singly exponential time into equivalent SNBA [5]. The translation in [5] is a generalization of the Miyano-Hayashi construction [30] (for converting by an exponential blowup a Büchi alternating automaton into an equivalent NBA) and exploits the so called *ranking* construction [25] for converting in quadratic time a coBüchi alternating automaton into an equivalent Büchi alternating automaton. The translation can be easily adapted for handling globally nondeterministic components in two-way HAA. Thus, we obtain the following result.

► **Proposition B.3** ([5]). *Given a two-way HAA \mathcal{A} , one can construct “on the fly” and in singly exponential time a Büchi SNBA accepting $\mathcal{L}_\varphi(\mathcal{A})$ with $2^{O(n \cdot \log(n+k))}$ states, where k is the number of states in the globally nondeterministic components of \mathcal{A} , and n is the number of remaining states of \mathcal{A} .*

B.1.2 Upper bounds of Theorem 4.1

In this section, in order to provide the upper bounds of Theorem 4.1, we first illustrate a compositional translation of a QPTL formula into an equivalent SNBA which improves the one given in [5]. There, occurrences of temporal modalities immediately preceding propositional quantification always count as additional alternations, and so they lead to an additional exponential blowup in the compositional translation. The relevant case in the translation is when the outermost operator of the currently processed sub-formula φ (assumed to be in negation normal form) is a temporal modality (the other cases easily follow from the closure of SNBA-definable pointed languages under union, intersection, and projection). This case is handled by first building a two-way HAA \mathcal{A} accepting $\mathcal{L}_\varphi(\varphi)$ and then by applying Proposition B.3. The construction of \mathcal{A} is obtained by a generalization of the standard linear-time translation of LTL formulas into Büchi alternating automata which exploits the (inductively built) SNBA associated with the maximal universal and existential sub-formulas of φ . Formally, we establish the following result, where for a QPTL formula φ , we say that φ is of *universal-type* if there is a universal sub-formula ψ of the negation normal form of φ such that $\text{sad}(\psi) = \text{sad}(\varphi)$; otherwise, we say that φ is of *existential-type*.

► **Theorem B.4.** *Let φ be a QPTL formula of existential-type (resp., of universal-type) and $h = \text{sad}(\varphi)$. Then, one can construct “on the fly” an SNBA \mathcal{A}_φ accepting $\mathcal{L}_\varphi(\varphi)$ in time $\text{Tower}(h + 1, O(|\varphi|))$ (resp., $\text{Tower}(h + 2, O(|\varphi|))$).*

Proof. We assume without loss of generality that φ is in negation normal form. The proof is given by induction on the structure of φ . The base case is trivial. For the induction step, we

distinguish four cases depending on the type of root operator of φ (either positive boolean connective, or existential quantifier, or universal quantifier, or temporal modality).

Case 1: φ is of the form $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi = \varphi_1 \vee \varphi_2$. Assume that $\varphi = \varphi_1 \wedge \varphi_2$ (the other case being similar). We use the fact that like Büchi nondeterministic automata, SNBA are trivially and efficiently closed under intersection. In particular, given two SNBA \mathcal{A}_1 and \mathcal{A}_2 , one can construct “on the fly” and in time $O(|\mathcal{A}_1||\mathcal{A}_2|)$ an SNBA accepting the language $\mathcal{L}_\varphi(\mathcal{A}_1) \cap \mathcal{L}_\varphi(\mathcal{A}_2)$. Since $\text{sad}(\varphi) = \max(\text{sad}(\varphi_1), \text{sad}(\varphi_2))$, the result easily follows from the induction hypothesis.

Case 2: φ is an existential formula of the form $\varphi = \exists p. \varphi'$. Hence, in particular, φ is of existential-type. Let $h = \text{sad}(\varphi)$ and $h' = \text{sad}(\varphi')$. We observe that like Büchi nondeterministic automata, SNBA are efficiently closed under projection. In particular, given an SNBA \mathcal{A} over traces and $p \in \text{AP}$, one can construct “on the fly” and in linear time an SNBA accepting the pointed language $\{(\sigma, i) \mid \text{there is } (\sigma', i) \in \mathcal{L}_\varphi(\mathcal{A}) \text{ such that } \sigma' =_{\text{AP} \setminus \{p\}} \sigma\}$. Thus, by applying the induction hypothesis, it follows that one can construct “on the fly” an SNBA accepting $\mathcal{L}_\varphi(\varphi)$ of size $\text{Tower}(h' + 1, O(|\varphi'|))$ if φ' is of existential-type, and of size $\text{Tower}(h' + 2, O(|\varphi'|))$ otherwise. Since $h' = h$ if φ' is of existential-type, and $h' = h - 1$ otherwise (i.e. φ' is of universal-type), the result follows.

Case 3: φ is a universal formula of the form $\varphi = \forall p. \varphi'$. Hence, in particular, φ is of universal-type. Let $h = \text{sad}(\varphi)$ and $\tilde{\varphi}'$ be the negation normal form of $\neg\varphi'$. We have that $\text{sad}(\exists p. \tilde{\varphi}') = h$ and $\mathcal{L}_\varphi(\exists p. \tilde{\varphi}') = \mathcal{L}_\varphi(\neg\varphi)$. Hence, by Case 2, one can construct “on the fly” an SNBA $\mathcal{A}_{\neg\varphi}$ of size $\text{Tower}(h + 1, O(|\varphi|))$ accepting $\mathcal{L}_\varphi(\neg\varphi)$. By Propositions B.1, B.2 and B.3, it follows that one can construct “on the fly” an SNBA \mathcal{A}_φ of size $\text{Tower}(h + 2, O(|\varphi|))$ accepting $\mathcal{L}_\varphi(\varphi)$. Hence, the result follows.

Case 4: the root operator of φ is a temporal modality. Let $h = \text{sad}(\varphi)$ and P be the set of existential and universal sub-formulas of φ which do not occur in the scope of a quantifier. If $P = \emptyset$, then φ is a PLTL formula and $h = 0$. In this case, by a straightforward adaptation of the standard translation of LTL into Büchi word automata [36], one can construct a SNBA of size $2^{O(|\varphi|)}$ accepting $\mathcal{L}_\varphi(\varphi)$. Hence, the result follows.

Assume now that $P \neq \emptyset$. Then, φ can be viewed as a PLTL formula in negation normal form, written $\text{PLTL}(\varphi)$, over the set of atomic propositions given by $P \cup \text{AP}$.

Subcase 4.1: we first assume that for each $\psi \in P$, $\text{sad}(\psi) < \text{sad}(\varphi)$. Hence, for all $\psi \in P$, $\text{sad}(\psi) \leq h - 1$ and $h \geq 1$. Note that φ is of existential-type. Moreover, for each universal formula $\forall p. \xi \in P$, we have that $\text{sad}(\forall p. \xi) = \text{sad}(\neg\exists p. \tilde{\xi})$ and $\mathcal{L}_\varphi(\forall p. \xi) = \mathcal{L}_\varphi(\neg\exists p. \tilde{\xi})$, where $\tilde{\xi}$ is the negation normal form of ξ . Thus, since each existential formula is of existential-type, by applying the induction hypothesis, Proposition B.2 and the complementation result for two-way HAA (see Proposition B.1), it follows that for each $\psi \in P$, one can construct “on the fly” in time at most $\text{Tower}(h, O(|\psi|))$, a two-way HAA \mathcal{A}_ψ accepting $\mathcal{L}_\varphi(\psi)$. Then, by an easy generalization of the standard linear-time translation of LTL formulas into Büchi alternating word automata and by using the two-way HAA \mathcal{A}_ψ with $\psi \in P$, we show that one can construct “on the fly”, in time $\text{Tower}(h, O(|\varphi|))$, a two-way HAA \mathcal{A}_φ accepting $\mathcal{L}_\varphi(\varphi)$. Intuitively, given an input pointed trace, each copy of \mathcal{A}_φ keeps track of the current subformula of $\text{PLTL}(\varphi)$ which needs to be evaluated. The evaluation simulates the semantics of PLTL by using universal and existential branching, but when the current subformula ψ is in P , then the current copy of \mathcal{A}_φ activates a copy of \mathcal{A}_ψ in the initial state.

Formally, for each $\psi \in P$, let $\mathcal{A}_\psi = \langle Q_\psi, q_\psi, \delta_\psi, F_\psi^-, F_\psi \rangle$. Without loss of generality, we assume that the state sets of the two-way \mathcal{A}_ψ are pairwise distinct. Then, $\mathcal{A}_\varphi =$

15:26 Unifying Asynchronous Logics for Hyperproperties

$\langle Q, q_0, \delta, F_-, F \rangle$, where:

- $Q = \bigcup_{\psi \in P} Q_\psi \cup \text{Sub}(\varphi)$, where $\text{Sub}(\varphi)$ is the set of subformulas of $\text{PLTL}(\varphi)$;
- $q_0 = \varphi$;
- The transition function δ is defined as follows: $\delta(q, a) = \delta_\psi(q, a)$ if $q \in Q_\psi$ for some $\psi \in P$. If instead $q \in \text{Sub}(\varphi)$, then $\delta(q, a)$ is inductively defined as follows:
 - $\delta(p, a) = \mathbf{true}$ if $p \in a$, and $\delta(p, a) = \mathbf{false}$ otherwise (for all $p \in \text{AP} \cap \text{Sub}(\varphi)$);
 - $\delta(\neg p, a) = \mathbf{false}$ if $p \in a$, and $\delta(\neg p, a) = \mathbf{true}$ otherwise (for all $p \in \text{AP} \cap \text{Sub}(\varphi)$);
 - $\delta(\phi_1 \wedge \phi_2, a) = \delta(\phi_1, a) \wedge \delta(\phi_2, a)$ and $\delta(\phi_1 \vee \phi_2, a) = \delta(\phi_1, a) \vee \delta(\phi_2, a)$;
 - $\delta(\mathbf{X}\phi, a) = (\rightarrow, \phi)$ and $\delta(\mathbf{Y}\phi, a) = (\leftarrow, \phi)$;
 - $\delta(\phi_1 \mathbf{U} \phi_2, a) = \delta(\phi_2, a) \vee (\delta(\phi_1, a) \wedge (\rightarrow, \phi_1 \mathbf{U} \phi_2))$;
 - $\delta(\phi_1 \mathbf{S} \phi_2, a) = \delta(\phi_2, a) \vee (\delta(\phi_1, a) \wedge (\leftarrow, \phi_1 \mathbf{S} \phi_2))$;
 - $\delta(\phi_1 \mathbf{R} \phi_2, a) = \delta(\phi_2, a) \wedge (\delta(\phi_1, a) \vee (\rightarrow, \phi_1 \mathbf{R} \phi_2))$;
 - $\delta(\phi_1 \mathbf{P} \phi_2, a) = \delta(\phi_2, a) \wedge (\delta(\phi_1, a) \vee (\leftarrow, \phi_1 \mathbf{P} \phi_2))$;
 - for each $\psi \in P$, $\delta(\psi, a) = \delta_\psi(q_\psi, \sigma)$.
- $F_- = \bigcup_{\psi \in P} F_\psi^-$
- $F = \bigcup_{\psi \in P} F_\psi \cup \bigcup_{\phi \in \text{Sub}(\varphi)} \{\mathcal{S}_\phi\}$, where for each $\phi \in \text{Sub}(\varphi)$, \mathcal{S}_ϕ is defined as follows:
 - if ϕ has as root a past temporal modality, then \mathcal{S}_ϕ is the negative stratum $(-, \{\phi\}, \emptyset)$;
 - if ϕ has as root the (future) until modality, then \mathcal{S}_ϕ is the Büchi stratum $(\mathbf{B}, \{\phi\}, \emptyset)$;
 - if ϕ has as root the (future) release modality, then \mathcal{S}_ϕ is the coBüchi stratum $(\mathbf{C}, \{\phi\}, \emptyset)$;
 - otherwise, \mathcal{S}_ϕ is the transient stratum given by $(\mathbf{t}, \{\phi\}, \emptyset)$.

Finally, since $h \geq 1$ and the size of the two-way HAA \mathcal{A}_φ is $\text{Tower}(h, O(|\varphi|))$, by applying Proposition B.3, one can construct “on the fly” an SNBA accepting $\mathcal{L}_\varphi(\varphi)$ of size $\text{Tower}(h + 1, O(|\varphi|))$. Hence, the result follows.

Subcase 4.2: for some $\psi \in P$, $\text{sad}(\psi) = \text{sad}(\varphi)$, and either $\varphi = \mathbf{X}\varphi_1$ or $\varphi = \mathbf{Y}\varphi_2$. This case easily follows from the induction hypothesis and the fact that given an SNBA \mathcal{A} one can easily construct in linear time two SNBA $\mathcal{A}_\mathbf{X}$ and $\mathcal{A}_\mathbf{Y}$ such that $\mathcal{L}_\varphi(\mathcal{A}_\mathbf{X}) = \{(\sigma, i) \mid (\sigma, i + 1) \in \mathcal{L}_\varphi(\mathcal{A})\}$ and $\mathcal{L}_\varphi(\mathcal{A}_\mathbf{Y}) = \{(\sigma, i) \mid i > 0 \text{ and } (\sigma, i - 1) \in \mathcal{L}_\varphi(\mathcal{A})\}$.

Subcase 4.3: for some $\psi \in P$, $\text{sad}(\psi) = \text{sad}(\varphi) = h$, and either $\varphi = \varphi_1 \mathbf{U} \varphi_2$ or $\varphi = \varphi_1 \mathbf{S} \varphi_2$. Let P_1 be the set of universal and existential sub-formulas of φ_1 . Since ψ is a universal or existential sub-formula of φ and $\text{sad}(\psi) = \text{sad}(\varphi)$, by definition of the strong alternation depth, it follows that (i) φ and φ_2 are of existential-type, (ii) $\text{sad}(\varphi_2) = \text{sad}(\varphi)$, and (iii) for each $\psi_1 \in P_1$, $\text{sad}(\psi_1) < h$. We focus on the case where $\varphi = \varphi_1 \mathbf{U} \varphi_2$ (the case where $\varphi = \varphi_1 \mathbf{S} \varphi_2$ is similar). By the induction hypothesis, one can construct an SNBA \mathcal{A}_2 in time $\text{Tower}(h + 1, O(|\varphi|))$ accepting $\mathcal{L}_\varphi(\varphi_2)$. We distinguish two cases:

- $h = 0$: hence, $P_1 = \emptyset$ and φ_1 is a PLTL formula. By an easy adaptation of the standard translation of LTL into Büchi word automata [36] and by exploiting the SNBA \mathcal{A}_2 for the formula φ_2 , one can construct an SNBA \mathcal{A}_φ of size $2^{O(|\varphi|)}$ accepting $\mathcal{L}_\varphi(\varphi)$. Intuitively, given an input pointed trace (σ, i) , \mathcal{A}_φ guesses a position $j \geq i$ and checks that $(\sigma, j) \in \mathcal{L}_\varphi(\mathcal{A})$ and for all $k \in [i, j)$, $(\sigma, k) \models \varphi_1$ as follows. Initially, \mathcal{A}_φ keeps track of both the guessed set Λ_0 of sub-formulas of φ_1 holding at the current position i , and the guessed state q of \mathcal{A}_2 which represents the state where the backward copy of \mathcal{A}_2 would be on reading the i^{th} position of σ in some guessed accepting run of \mathcal{A}_2 over (σ, j) . If $j = i$, then q needs to be some initial state of \mathcal{A}_2 , and \mathcal{A}_2 simply simulates the behavior of \mathcal{A}_2 on (σ, i) and propagates the guesses about the sub-formulas of φ_1 in accordance to the semantics of PLTL. Otherwise, \mathcal{A}_φ splits in two copies: the backward copy simulates

the backward copy of \mathcal{A}_2 and *deterministically* checks that the initial guessed set Γ_0 of sub-formulas of φ_1 contains φ_1 and is consistent in the interval of positions $[0, i]$, while the forward copy of \mathcal{A}_φ behaves as follows. In the first step, the forward copy of \mathcal{A}_φ moves to the same state (q, Γ_0) , and after this step, such a copy starts to simulate in forward-mode the backward copy of \mathcal{A}_2 until, possibly, a ‘switch’ occurs at the guessed position j , where the forward copy of \mathcal{A}_φ simulates in a unique step from the current state some initial split of \mathcal{A}_2 in the backward and forward copy. In the phase before the switch, the current guessed set of sub-formulas always contains φ_1 . After such a switch (if any), the forward copy of \mathcal{A}_φ simply simulates the forward copy of \mathcal{A}_2 and propagates the guesses about the sub-formulas of φ_1 in accordance to the semantics of PLTL. We use two flags to distinguish the different phases of the simulation (in particular, the initial phase and the switch phase). The acceptance condition is a generalized Büchi condition which can be converted into a Büchi condition in a standard way.

- $h \geq 1$: in this case, we construct a two-way HAA \mathcal{A}_φ accepting $\mathcal{L}_\varphi(\varphi)$ as done for the subcase 4.1 but we replace the set P with the set $P_1 \cup \{\varphi_2\}$. Note that for each $\psi \in P_1$, being $sad(\psi) < h$, the two-way HAA associated with formula ψ has size at most $\text{Tower}(h, O(|\psi|))$. Moreover, let \mathcal{A}_{φ_2} be the two-way HAA associated with the SNBA \mathcal{A}_2 accepting $\mathcal{L}_\varphi(\varphi_2)$. \mathcal{A}_{φ_2} has size at most $\text{Tower}(h + 1, O(|\varphi_2|))$ and since it is the two-way HAA associated to an SNBA, it has just two strata: a negative *existential* stratum, say \mathcal{S}_- , and a Büchi stratum (a Büchi stratum is always existential), say \mathcal{S}_B . Thus, applying the construction illustrated for the subcase 4.1, we have that the upper stratum \mathcal{S}_φ of the two-way HAA \mathcal{A}_φ associated with the formula $\varphi = \varphi_1 \mathbf{U} \varphi_2$ is a Büchi stratum consisting of the single state φ which is the initial state. Additionally, \mathcal{S}_φ is the unique stratum from which it is possible to move to the strata of \mathcal{A}_{φ_2} , and for all input symbols a , if $\delta_\varphi(\varphi, a)$ is rewritten in disjunctive normal form, then each disjunction contains at most one state from \mathcal{S}_- and at most one state from \mathcal{S}_B . In other terms, the strata \mathcal{S}_- and \mathcal{S}_B are globally nondeterministic in \mathcal{A}_φ . Thus, \mathcal{A}_φ has at most $\text{Tower}(h + 1, O(|\varphi_2|))$ globally nondeterministic states while the number of remaining states is at most $\text{Tower}(h, O(|\varphi_1|))$. By applying Proposition B.3, one can construct “on the fly” an SNBA accepting $\mathcal{L}_\varphi(\varphi)$ of size $\text{Tower}(h + 1, O(|\varphi|))$. Hence, the result follows.

Subcase 4.4: for some $\psi \in P$, $sad(\psi) = sad(\varphi)$, and either $\varphi = \varphi_1 \mathbf{R} \varphi_2$ or $\varphi = \varphi_1 \mathbf{P} \varphi_2$. Since ψ is a universal or existential sub-formula of φ_1 or φ_2 , by definition of $sad(\varphi)$, ψ must be a universal formula. Hence, φ is of universal-type. Let $\tilde{\varphi}'$ be the negation normal form of $\neg\varphi'$. We have that $sad(\tilde{\varphi}') = h$ and $\mathcal{L}_\varphi(\tilde{\varphi}') = \mathcal{L}_\varphi(\neg\varphi)$. Hence, by the subcase 4.3, one can construct “on the fly” an SNBA $\mathcal{A}_{\neg\varphi}$ of size $\text{Tower}(h + 1, O(|\varphi|))$ accepting $\mathcal{L}_\varphi(\neg\varphi)$. By Propositions B.1, B.2 and B.3, it follows that one can construct “on the fly” an SNBA \mathcal{A}_φ of size $\text{Tower}(h + 2, O(|\varphi|))$ accepting $\mathcal{L}_\varphi(\varphi)$. Hence, the result follows.

This concludes the proof of Theorem B.4. ◀

By exploiting Theorem B.4, we can provide the upper bounds of Theorem 4.1.

► **Theorem B.5.** *For all $h \geq 0$, satisfiability of QPTL sentences φ with strong alternation depth at most h is in h -EXPSPACE.*

Proof. We observe that a QPTL sentence is satisfiable iff it is valid. Thus, since a QPTL sentence in negation normal form is a positive Boolean combination of universal and existential sentences, it suffices to show the result for existential and universal QPTL sentences φ :

- $\varphi = \exists p. \varphi'$: hence, φ is of existential-type. By Theorem B.4, one can construct “on the fly” an SNBA \mathcal{A}_φ accepting $\mathcal{L}_\varphi(\varphi)$ in time $\text{Tower}(h + 1, O(|\varphi|))$. We observe that an

SNBA \mathcal{A} can be trivially converted into a Büchi nondeterministic automaton accepting the set of traces σ such that $(\sigma, 0) \in \mathcal{L}_\varphi(\mathcal{A})$. Thus, since checking non-emptiness for Büchi nondeterministic automata is in NLOGSPACE, the result follows.

- $\varphi = \forall p. \varphi'$: since φ is a sentence, we have that φ is satisfiable iff $\exists p. \neg\varphi$ is unsatisfiable. Thus, since $\text{sad}(\varphi) = \text{sad}(\exists p. \neg\varphi)$, this case reduces to the previous case. ◀

B.2 Detailed proof of Theorem 4.2

For a QPTL formula φ and $\text{AP}' \subseteq \text{AP}$ with $\text{AP}' = \{p_1, \dots, p_n\}$, we write $\exists \text{AP}'.\varphi$ to mean $\exists p_1. \dots \exists p_n. \varphi$. Given a fair Kripke structure (\mathcal{K}, F) , a (\mathcal{K}, F) -assignment is a partial mapping Π over VAR assigning to each trace variable x in its domain $\text{Dom}(x)$ a pair (π, i) consisting of a F -fair path of \mathcal{K} and a position $i \geq 0$.

► **Theorem 4.2.** *Given a fair finite Kripke structure (\mathcal{K}, F) and a SHyperLTL $_{S+C}^\emptyset$ sentence φ , one can construct in linear time a QPTL sentence ψ with the same strong alternation depth as φ such that ψ is satisfiable if and only if $\mathcal{L}(\mathcal{K}, F) \models \varphi$.*

Proof. Let $\mathcal{K} = \langle S, S_0, E, \text{Lab} \rangle$. The high-level description of the reduction of model checking (\mathcal{K}, F) against φ to QPTL satisfiability has been given in Section 4. Here, we provide the details of the reduction. We consider a new finite set AP' of atomic propositions defined as follows:

$$\begin{aligned} \text{AP}' &:= \bigcup_{x \in \text{VAR}} \text{AP}_x \cup S_x \cup \{\#_{\overleftarrow{x}}, \#_{\overrightarrow{x}}\} \\ \text{AP}_x &:= \{p_{\overleftarrow{x}}, p_{\overrightarrow{x}} \mid p \in \text{AP}\} \text{ and } S_x := \{s_{\overleftarrow{x}}, s_{\overrightarrow{x}} \mid s \in S\} \end{aligned}$$

Let $\text{AP}_{\overleftarrow{x}} := \{p_{\overleftarrow{x}} \mid p \in \text{AP}\}$, $\text{AP}_{\overrightarrow{x}} := \{p_{\overrightarrow{x}} \mid p \in \text{AP}\}$, $S_{\overleftarrow{x}} := \{s_{\overleftarrow{x}} \mid s \in S\}$, and $S_{\overrightarrow{x}} := \{s_{\overrightarrow{x}} \mid s \in S\}$. Thus, we associate to each variable $x \in \text{VAR}$ and atomic proposition $p \in \text{AP}$, two fresh atomic propositions $p_{\overleftarrow{x}}$ and $p_{\overrightarrow{x}}$, and to each variable $x \in \text{VAR}$ and state s of \mathcal{K} , two fresh atomic proposition $s_{\overleftarrow{x}}$ and $s_{\overrightarrow{x}}$. Moreover, for each $x \in \text{VAR}$, $\#_{\overleftarrow{x}}$ and $\#_{\overrightarrow{x}}$ are exploited as padding propositions.

Encoding of paths and pointed paths of \mathcal{K} . For each $x \in \text{VAR}$ and an infinite word $\pi = s_0, s_1, \dots$ over S , we denote by $\overrightarrow{\sigma}(x, \pi)$ the trace over $\text{AP}_{\overrightarrow{x}} \cup S_{\overrightarrow{x}}$ defined as follows for all $i \geq 0$:

$$\overrightarrow{\sigma}(x, \pi)(i) := \{(s_i)_{\overrightarrow{x}}\} \cup \{p_{\overrightarrow{x}} \mid p \in \text{Lab}(s_i)\}$$

For each finite word π over S , let $\overleftarrow{\sigma}(x, \pi)$ be the trace over $\text{AP}_{\overleftarrow{x}} \cup S_{\overleftarrow{x}}$ given by $\{\#_{\overleftarrow{x}}\} \cdot w \cdot \{\#_{\overleftarrow{x}}\}^\omega$, where $|w| = |\pi|$ and for each $0 \leq i < |\pi|$, $w(i) = \{(s_i)_{\overleftarrow{x}}\} \cup \{p_{\overleftarrow{x}} \mid p \in \text{Lab}(s_i)\}$. Now, we define the forward and backward encodings of a path π of \mathcal{K} . For each $k \in \mathbb{N}$, the *forward x -encoding of π with offset k* is the trace over $\text{AP}_x \cup S_x \cup \{\#_{\overleftarrow{x}}, \#_{\overrightarrow{x}}\}$, denoted by $\overrightarrow{\sigma}(x, \pi, k)$, defined as follows:

- the projection of $\overrightarrow{\sigma}(x, \pi, k)$ over $\text{AP}_{\overrightarrow{x}} \cup S_{\overrightarrow{x}} \cup \{\#_{\overrightarrow{x}}\}$ is $\{\#_{\overrightarrow{x}}\}^k \cdot \overrightarrow{\sigma}(x, \pi)$;
- the projection of $\overrightarrow{\sigma}(x, \pi, k)$ over $\text{AP}_{\overleftarrow{x}} \cup S_{\overleftarrow{x}} \cup \{\#_{\overleftarrow{x}}\}$ is $\{\#_{\overleftarrow{x}}\}^\omega$.

For each $k > 0$, the *backward x -encoding of π with offset $k > 0$* , is the trace over $\text{AP}_x \cup S_x \cup \{\#_{\overleftarrow{x}}, \#_{\overrightarrow{x}}\}$, denoted by $\overleftarrow{\sigma}(x, \pi, k)$, defined as follows, where $\pi_{\leq k}^R$ denotes the reverse of the prefix $\pi(0) \dots \pi(k-1)$ of π until position $k-1$:

- the projection of $\overrightarrow{\sigma}(x, \pi, k)$ over $\text{AP}_{\overrightarrow{x}} \cup S_{\overrightarrow{x}} \cup \{\#_{\overrightarrow{x}}\}$ is $\overrightarrow{\sigma}(x, \pi^k)$;
- the projection of $\overrightarrow{\sigma}(x, \pi, k)$ over $\text{AP}_{\overleftarrow{x}} \cup S_{\overleftarrow{x}} \cup \{\#_{\overleftarrow{x}}\}$ is $\overleftarrow{\sigma}(x, \pi_{\leq k}^R)$.

Claim 1. For all $x \in \text{VAR}$, one can construct in linear time two PLTL formulas $\theta(x, \rightarrow)$ and $\theta(x, \leftarrow)$ such that for all pointed traces (σ, i) over AP' , we have:

- $(\sigma, i) \models \theta(x, \rightarrow)$ iff there is a F -fair path π of \mathcal{K} such that the projection of σ over $S_x \cup \text{AP}_x \cup \{\#_{\overleftarrow{x}}, \#_{\overrightarrow{x}}\}$ is the forward x -encoding of π for some offset $k \geq 0$.
- $(\sigma, i) \models \theta(x, \leftarrow)$ iff there is a F -fair path π of \mathcal{K} such that the projection of σ over $S_x \cup \text{AP}_x \cup \{\#_{\overleftarrow{x}}, \#_{\overrightarrow{x}}\}$ is the backward x -encoding of π for some offset $k > 0$.

We give the details on the construction of the PLTL formula $\theta(x, \leftarrow)$ in Claim 1 (the construction of $\theta(x, \rightarrow)$ is similar). For each state $s \in S$, $E(s)$ denotes the set of successors of s in \mathcal{K} , while $E^{-1}(s)$ denotes the set of predecessors of s in \mathcal{K} . The PLTL formula $\theta(x, \leftarrow)$ is defined as follows:

$$\begin{aligned} \theta(x, \leftarrow) &:= \mathbf{O}((\neg \mathbf{Y} \top) \wedge \#_{\overleftarrow{x}} \wedge \mathbf{G} \neg \#_{\overrightarrow{x}} \wedge \bigvee_{s \in S} [\xi(x, s, \rightarrow) \wedge \bigvee_{s' \in E^{-1}(s)} \mathbf{X} \xi(x, s', \leftarrow)]) \\ \xi(x, s, \rightarrow) &:= s_{\overrightarrow{x}} \wedge \bigvee_{s' \in F} \mathbf{G} \mathbf{F} s'_{\overrightarrow{x}} \wedge \bigwedge_{s' \in S} \mathbf{G} \left(s'_{\overrightarrow{x}} \rightarrow \right. \\ &\quad \left. \left[\bigwedge_{p \in \text{Lab}(s')} p_{\overrightarrow{x}} \wedge \bigwedge_{p \in \text{AP} \setminus \text{Lab}(s')} \neg p_{\overrightarrow{x}} \wedge \bigwedge_{s'' \in S \setminus \{s'\}} \neg s''_{\overrightarrow{x}} \wedge \bigvee_{s'' \in E(s')} \mathbf{X} s''_{\overrightarrow{x}} \right] \right) \\ \xi(x, s, \leftarrow) &:= s_{\overleftarrow{x}} \wedge \mathbf{F} \mathbf{G} \#_{\overleftarrow{x}} \wedge \bigvee_{s' \in S_0} \mathbf{F} (s'_{\overleftarrow{x}} \wedge \mathbf{X} \#_{\overleftarrow{x}}) \wedge \mathbf{H} \mathbf{G} [\#_{\overleftarrow{x}} \rightarrow \bigwedge_{q \in \text{AP} \cup S} \neg q_{\overleftarrow{x}}] \wedge \\ &\quad \bigwedge_{s' \in S} \mathbf{G} \left(s'_{\overleftarrow{x}} \rightarrow \left[\bigwedge_{p \in \text{Lab}(s')} p_{\overleftarrow{x}} \wedge \bigwedge_{p \in \text{AP} \setminus \text{Lab}(s')} \neg p_{\overleftarrow{x}} \wedge \neg \#_{\overleftarrow{x}} \wedge \right. \right. \\ &\quad \left. \left. \bigwedge_{s'' \in S \setminus \{s'\}} \neg s''_{\overleftarrow{x}} \wedge \mathbf{X} (\#_{\overleftarrow{x}} \vee \bigvee_{s'' \in E^{-1}(s')} s''_{\overleftarrow{x}}) \right] \right) \end{aligned}$$

We now extend the notions of backward and forward encodings of paths of \mathcal{K} to pointed paths (π, i) of \mathcal{K} . For each $k \in \mathbb{N}$, the *forward x -encoding of (π, i) with offset k* is the pointed trace given by $(\overrightarrow{\sigma}(x, \pi, k), i + k)$. We say that the position $i + k$ is in *forward mode* in the encoding. For each $k > 0$, the *backward x -encoding of (π, i) with offset k* is the pointed trace given by $(\overleftarrow{\sigma}(x, \pi, k), j)$, where $j = i - k$ if $i \geq k$, and $j = k - i$ otherwise. In the first case, we say that position j is in *forward mode* in the encoding, and the second case, we say that position j is in *backward mode* in the encoding. Intuitively, when (σ, ℓ) is a backward or forward encoding of a pointed path (π, i) of \mathcal{K} , then ℓ represents the position in the encoding associated to position i of π .

Encoding of (\mathcal{K}, F) -path assignments. Let Π be a (\mathcal{K}, F) -path assignment. A pointed trace (σ, i) over AP' is a *forward encoding of Π* if for each $x \in \text{VAR}$, the following holds, where σ_x denotes the projection of σ over $S_x \cup \text{AP}_x \cup \{\#_{\overleftarrow{x}}, \#_{\overrightarrow{x}}\}$:

- if $x \notin \text{Dom}(\Pi)$, then σ_x is \emptyset^ω ;
- if $x \in \text{Dom}(\Pi)$, then (σ_x, i) is a forward or backward encoding of $\Pi(x)$ where i is in forward mode in the encoding.

When $\text{Dom}(\Pi) \neq \emptyset$, we also consider the notion of a *backward encoding (σ, i)* of Π which is defined as a forward encoding but we require that for each $x \in \text{Dom}(\Pi)$, (σ_x, i) is a backward encoding of $\Pi(x)$ where i is in backward mode in the encoding. Note that in this case, we have that $i \geq 1$ holds.

Let Π be a (\mathcal{K}, F) -path assignment and Π' be the trace assignment over $\mathcal{L}(\mathcal{K}, F)$ obtained from Π in the obvious way. For each $\ell \geq 0$, we write $\text{succ}^\ell(\Pi)$ to mean $\text{succ}_{(\emptyset, \text{VAR})}^\ell(\Pi')$ and $\text{pred}^\ell(\Pi)$ to mean $\text{pred}_{(\emptyset, \text{VAR})}^\ell(\Pi')$. By construction, we easily obtain the following result, where $\text{Halt}_{\rightarrow} := \bigvee_{x \in \text{VAR}} \#_{\overrightarrow{x}}$ and $\text{Halt}_{\leftarrow} := \bigvee_{x \in \text{VAR}} \#_{\overleftarrow{x}}$.

Claim 2. Let Π be a (\mathcal{K}, F) -path assignment and $\ell \geq 0$.

- If (σ, i) is a forward encoding of Π , then:

15:30 Unifying Asynchronous Logics for Hyperproperties

- $(\sigma, i + \ell)$ is a forward encoding of $\text{succ}^\ell(\Pi)$;
- if $\ell \leq i$, then $\text{pred}^\ell(\Pi) \neq \text{und}$ iff $(\sigma, i - \ell) \models \neg \text{Halt}_{\rightarrow}$. Moreover, if $\text{pred}^\ell(\Pi) \neq \text{und}$, then $(\sigma, i - \ell)$ is a forward encoding of $\text{pred}^\ell(\Pi)$;
- if $\ell > i$, then $\text{pred}^\ell(\Pi) \neq \text{und}$ iff $(\sigma, \ell - i) \models \neg \text{Halt}_{\leftarrow}$. Moreover, if $\text{pred}^\ell(\Pi) \neq \text{und}$, then $(\sigma, \ell - i)$ is a backward encoding of $\text{pred}^\ell(\Pi)$.
- If (σ, i) is a backward encoding of Π , then:
 - If $\ell < i$, $(\sigma, i - \ell)$ is a backward encoding of $\text{succ}^\ell(\Pi)$;
 - if $\ell \geq i$, $(\sigma, \ell - i)$ is a forward encoding of $\text{succ}^\ell(\Pi)$;
 - $\text{pred}^\ell(\Pi) \neq \text{und}$ iff $(\sigma, i + \ell) \models \neg \text{Halt}_{\leftarrow}$. Moreover, if $\text{pred}^\ell(\Pi) \neq \text{und}$, then $(\sigma, i + \ell)$ is a backward encoding of $\text{pred}^\ell(\Pi)$.

Reduction to QPTL satisfiability. We define by structural induction a mapping $\mathbb{T} : \{\leftarrow, \rightarrow\} \times \text{SHyperLTL}_{\mathcal{S}+\mathcal{C}}^\emptyset \rightarrow \text{QPTL}$ associating to each pair (dir, ϕ) consisting of a direction $dir \in \{\leftarrow, \rightarrow\}$ and a $\text{SHyperLTL}_{\mathcal{S}+\mathcal{C}}^\emptyset$ formula ϕ over AP and VAR a QPTL formula $\mathbb{T}(dir, \phi)$ over AP' . Define $\text{AP}'_x := \text{AP}_x \cup \mathcal{S}_x \cup \{\#_{\overleftarrow{x}}, \#_{\overrightarrow{x}}\}$.

- $\mathbb{T}(dir, \top) = \top$;
- $\mathbb{T}(\leftarrow, p[x]) = p_{\overleftarrow{x}}$ for all $p \in \text{AP}$;
- $\mathbb{T}(\rightarrow, p[x]) = p_{\overrightarrow{x}}$ for all $p \in \text{AP}$;
- $\mathbb{T}(dir, \langle x \rangle \psi[x]) = \mathbb{T}_x(dir, \psi[x])$, where $\mathbb{T}_x(dir, \psi[x])$ is obtained from $\mathbb{T}(dir, \psi[x])$ by replacing each occurrence of Halt_{\leftarrow} (resp., $\text{Halt}_{\rightarrow}$) with $\#_{\overleftarrow{x}}$ (resp., $\#_{\overrightarrow{x}}$);
- $\mathbb{T}(dir, \neg \phi) = \neg \mathbb{T}(dir, \phi)$;
- $\mathbb{T}(dir, \phi_1 \vee \phi_2) = \mathbb{T}(dir, \phi_1) \vee \mathbb{T}(dir, \phi_2)$;
- $\mathbb{T}(\leftarrow, \mathbf{X}\phi) = \mathbf{Y}(\mathbb{T}(\leftarrow, \phi) \wedge \mathbf{Y}\top) \vee \mathbf{Y}(\mathbb{T}(\rightarrow, \phi) \wedge \neg \mathbf{Y}\top)$;
- $\mathbb{T}(\rightarrow, \mathbf{X}\phi) = \mathbf{X}\mathbb{T}(\rightarrow, \phi)$;
- $\mathbb{T}(\leftarrow, \mathbf{Y}\phi) = \mathbf{X}(\mathbb{T}(\leftarrow, \phi) \wedge \neg \text{Halt}_{\leftarrow})$;
- $\mathbb{T}(\rightarrow, \mathbf{Y}\phi) = \mathbf{Y}(\mathbb{T}(\rightarrow, \phi) \wedge \neg \text{Halt}_{\rightarrow}) \vee [\neg \mathbf{Y}\top \wedge \mathbf{X}(\mathbb{T}(\leftarrow, \phi) \wedge \neg \text{Halt}_{\leftarrow})]$;
- $\mathbb{T}(\leftarrow, \phi_1 \mathbf{U} \phi_2) = \mathbb{T}(\leftarrow, \phi_1) \mathbf{S}(\mathbf{Y}\top \wedge \mathbb{T}(\leftarrow, \phi_2)) \vee [\mathbf{H}(\mathbf{Y}\top \rightarrow \mathbb{T}(\leftarrow, \phi_1)) \wedge \mathbf{O}(\neg \mathbf{Y}\top \wedge \mathbb{T}(\rightarrow, \phi_1) \mathbf{U} \mathbb{T}(\rightarrow, \phi_2))]$;
- $\mathbb{T}(\rightarrow, \phi_1 \mathbf{U} \phi_2) = \mathbb{T}(\rightarrow, \phi_1) \mathbf{U} \mathbb{T}(\rightarrow, \phi_2)$;
- $\mathbb{T}(\leftarrow, \phi_1 \mathbf{S} \phi_2) = \mathbb{T}(\leftarrow, \phi_1) \mathbf{U}(\mathbb{T}(\leftarrow, \phi_2) \wedge \neg \text{Halt}_{\leftarrow})$;
- $\mathbb{T}(\rightarrow, \phi_1 \mathbf{S} \phi_2) = \mathbb{T}(\rightarrow, \phi_1) \mathbf{S}(\mathbb{T}(\rightarrow, \phi_2) \wedge \neg \text{Halt}_{\rightarrow}) \vee [\mathbf{H}\mathbb{T}(\rightarrow, \phi_1) \wedge \mathbf{O}(\neg \mathbf{Y}\top \wedge \mathbf{X}(\mathbb{T}(\leftarrow, \phi_1) \mathbf{U}(\mathbb{T}(\leftarrow, \phi_2) \wedge \neg \text{Halt}_{\leftarrow})))]$;
- $\mathbb{T}(\leftarrow, \exists x. \phi) = \exists \text{AP}'_x. (\theta(x, \leftarrow) \wedge \mathbb{T}(\leftarrow, \phi) \wedge \neg \#_{\overleftarrow{x}} \wedge \mathbf{X}\#_{\overleftarrow{x}})$;
- $\mathbb{T}(\rightarrow, \exists x. \phi) = \exists \text{AP}'_x. (\theta(x, \rightarrow) \wedge \mathbb{T}(\rightarrow, \phi) \wedge \neg \#_{\overrightarrow{x}} \wedge (\mathbf{Y}\top \rightarrow \mathbf{Y}\#_{\overrightarrow{x}}))$;
- $\mathbb{T}(\leftarrow, \exists^P x. \phi) = \exists \text{AP}'_x. (\theta(x, \leftarrow) \wedge \mathbb{T}(\leftarrow, \phi) \wedge \neg \#_{\overleftarrow{x}})$;
- $\mathbb{T}(\rightarrow, \exists^P x. \phi) = \exists \text{AP}'_x. (\mathbb{T}(\rightarrow, \phi) \wedge [\theta(x, \leftarrow) \vee (\theta(x, \rightarrow) \wedge \neg \#_{\overrightarrow{x}})])$;

where $\theta(x, \leftarrow)$ and $\theta(x, \rightarrow)$ are the PLTL formulas of Claim 1. By construction $\mathbb{T}(dir, \phi)$ has size linear in ϕ and has the same strong alternation depth as ϕ . Moreover, $\mathbb{T}(dir, \phi)$ is a QPTL sentence if ϕ is a $\text{SHyperLTL}_{\mathcal{S}+\mathcal{C}}^\emptyset$ sentence. Now, we prove that the construction is correct. Given a $\text{SHyperLTL}_{\mathcal{S}+\mathcal{C}}^\emptyset$ formula ϕ and a (\mathcal{K}, F) -assignment Π such that $\text{Dom}(\Pi)$ consists of all and only the trace variables occurring free in ϕ , we write $\Pi \models \phi$ to mean that $(\Pi', \text{VAR}) \models_{\mathcal{L}(\mathcal{K}, F)} \phi$, where Π' is the trace assignment over $\mathcal{L}(\mathcal{K}, F)$ obtained from Π in the obvious way. For a trace σ over AP' , a variable $x \in \text{VAR}$, and a trace σ_x over AP'_x , we denote by $\sigma[x \mapsto \sigma_x]$ the trace σ' over AP' such that $\sigma' =_{\text{AP}' \setminus \text{AP}'_x} \sigma$ and the projection of σ' over AP'_x is σ_x . For a $\text{SHyperLTL}_{\mathcal{S}+\mathcal{C}}^\emptyset$ sentence φ , we show that $\Pi_\emptyset \models \varphi \Leftrightarrow (\emptyset^\omega, 0) \models \mathbb{T}(\rightarrow, \varphi)$ where $\text{Dom}(\Pi_\emptyset) = \emptyset$. The result directly follows from the following claim.

Claim 3: let $dir \in \{\leftarrow, \rightarrow\}$, ϕ be a $\text{SHyperLTL}_{\mathcal{S}+\mathcal{C}}^\emptyset$ formula, and Π be a (\mathcal{K}, F) -assignment such that $\text{Dom}(\Pi)$ contains all the trace variables occurring free in ϕ and $dir \Rightarrow$ if $\text{Dom}(\Pi) = \emptyset$.

Then, for all encodings (σ, i) of Π such that (σ, i) is a forward encoding if $dir = \rightarrow$, and a backward encoding otherwise, the following holds: $\Pi \models \phi \Leftrightarrow (\sigma, i) \models \mathsf{T}(dir, \phi)$.

The proof of Claim 3 is by structural induction on ϕ . When ϕ is a trace-relativized atomic proposition, the result directly follows from the definition of the map T and the notion of an encoding (σ, i) of Π . The cases for the boolean connectives directly follow from the induction hypothesis, while the cases relative to the temporal modalities easily follow from the induction hypothesis, Claim 2, and the definition of the map T . For the other cases, the ones relative to the hyper quantifiers, we proceed as follows:

- $\phi = \exists x. \phi'$: we focus on the case, where $dir = \leftarrow$ (the other case being similar). By hypothesis, (σ, i) is a backward encoding of Π and $Dom(\Pi) \neq \emptyset$. Hence, it holds that $i \geq 1$. For the implication $\Pi \models \phi \Rightarrow (\sigma, i) \models \mathsf{T}(\leftarrow, \phi)$, assume that $\Pi \models \phi$. Hence, there exists a F -fair path π of \mathcal{K} such that $\Pi[x \mapsto (\pi, 0)] \models \phi'$. Since $i \geq 1$, by construction, the trace σ_x over AP'_x given by $(\overleftarrow{\sigma}(x, \pi, i), i)$ is the backward x -encoding of $(\pi, 0)$ where i is in backward mode in the encoding. Hence, the trace σ' given by $\sigma[x \mapsto \sigma_x]$ is a backward encoding of $\Pi[x \mapsto (\pi, 0)]$. By the induction hypothesis, $(\sigma', i) \models \mathsf{T}(\leftarrow, \phi')$. Moreover, by construction and Claim 1, $(\sigma_x, i) \models \theta(x, \leftarrow)$, $\#_{\overleftarrow{x}} \notin \sigma_x(i)$ and $\#_{\overleftarrow{x}} \in \sigma_x(i+1)$. Hence, by definition of $\mathsf{T}(\leftarrow, \exists x. \phi')$, we obtain that $(\sigma, i) \models \mathsf{T}(\leftarrow, \phi)$.

For the converse implication, assume that $(\sigma, i) \models \mathsf{T}(\leftarrow, \phi)$. By definition of $\mathsf{T}(\leftarrow, \exists x. \phi')$ and Claim 1, there exists a trace σ_x over AP'_x such that σ_x is a backward x -encoding $\overleftarrow{\sigma}(x, \pi, k)$ of some F -fair path π of \mathcal{K} for some offset $k > 0$. Moreover, $(\sigma', i) \models \mathsf{T}(\leftarrow, \phi')$, where σ' is given by $\sigma[x \mapsto \sigma_x]$, $\#_{\overleftarrow{x}} \notin \sigma_x(i)$ and $\#_{\overleftarrow{x}} \notin \sigma_x(i+1)$. This means that the offset k is exactly i . Hence, $(\overleftarrow{\sigma}(x, \pi, i), i)$ is a backward x -encoding of $(\pi, 0)$ where i is in backward mode in the encoding, and (σ', i) is a backward encoding of $\Pi[x \mapsto (\pi, 0)]$. Thus, by the induction hypothesis, the result directly follows.

- $\phi = \exists^P x. \phi'$: we focus on the case, where $dir = \rightarrow$ (the other case being similar). By hypothesis, (σ, i) is a forward encoding of Π . For the implication $\Pi \models \phi \Rightarrow (\sigma, i) \models \mathsf{T}(\rightarrow, \phi)$, assume that $\Pi \models \phi$. Hence, there exists a F -fair path π of \mathcal{K} and a position $\ell \geq 0$ such that $\Pi[x \mapsto (\pi, \ell)] \models \phi'$. Let (σ_x, i) be the trace over AP'_x defined as follows:
 - if $\ell \leq i$: we set (σ_x, i) to $(\overrightarrow{\sigma}(x, \pi, i - \ell), i)$ which by construction is a forward x -encoding of (π, ℓ) . By construction and Claim 1, $(\sigma_x, i) \models \theta(x, \rightarrow) \wedge \neg \#_{\overrightarrow{x}}$;
 - if $\ell > i$: we set (σ_x, i) to $(\overleftarrow{\sigma}(x, \pi, \ell - i), i)$ which by construction is a backward x -encoding of (π, ℓ) where i is in forward mode in the encoding. By construction and Claim 1, $(\sigma_x, i) \models \theta(x, \leftarrow)$.

Hence, the trace σ' given by $\sigma[x \mapsto \sigma_x]$ is a forward encoding of $\Pi[x \mapsto (\pi, \ell)]$. By the induction hypothesis, $(\sigma', i) \models \mathsf{T}(\leftarrow, \phi')$. Hence, by definition of $\mathsf{T}(\rightarrow, \exists^P x. \phi')$, we obtain that $(\sigma, i) \models \mathsf{T}(\rightarrow, \phi)$.

For the converse implication, assume that $(\sigma, i) \models \mathsf{T}(\rightarrow, \phi)$. By definition of $\mathsf{T}(\rightarrow, \exists x. \phi')$ and Claim 1, there exists a F -fair path π of \mathcal{K} and a trace σ_x over AP'_x such that $(\sigma', i) \models \mathsf{T}(\rightarrow, \phi')$, where σ' is given by $\sigma[x \mapsto \sigma_x]$ and one of the following holds:

- σ_x is the forward x -encoding $\overrightarrow{\sigma}(x, \pi, k)$ of π for some offset $k \geq 0$ and $\#_{\overrightarrow{x}} \notin \sigma_x(i)$. It follows that $k \leq i$ and (σ_x, i) is the forward x -encoding of the pointed path $(\pi, i - k)$.
- σ_x is the backward x -encoding $\overleftarrow{\sigma}(x, \pi, k)$ of π for some offset $k > 0$. Hence, (σ_x, i) is the backward x -encoding of the pointed path $(\pi, i + k)$ where i is in forward mode in the encoding.

Hence, for some $\ell \geq 0$, (σ', i) is a forward encoding of $\Pi[x \mapsto (\pi, \ell)]$. Thus, being $(\sigma', i) \models \mathsf{T}(\rightarrow, \phi')$, by the induction hypothesis, the result directly follows.

This concludes the proof of Claim 3 and Theorem 4.2 too. ◀

B.3 Proof of Theorem 4.3

► **Theorem 4.3.** *Given a QPTL sentence ψ over AP , one can build in linear time a finite Kripke structure \mathcal{K}_{AP} (depending only on AP) and a singleton-free SHyperLTL $_{S+C}^{\emptyset}$ sentence φ having the same strong alternation depth as ψ such that ψ is satisfiable iff $\mathcal{L}(\mathcal{K}_{AP}) \models \varphi$.*

Proof. Without loss of generality, we only consider *well-named* QPTL formulas, i.e. QPTL formulas where each quantifier introduces a different proposition. Moreover, we can assume that AP is the set of all and only the propositions occurring in the given QPTL sentence. Let $AP' = AP \cup \{tag, in\}$, where *tag* and *in* are fresh propositions, and fix an ordering $\{p_1, \dots, p_n\}$ of the propositions in AP . First, we encode a trace σ over AP by a trace $en(\sigma)$ over AP' defined as follows: $en(\sigma) := \sigma_0 \cdot \sigma_1 \cdot \dots$, where for each $i \geq 0$, σ_i (the encoding of the i^{th} symbol of σ) is the finite word over $2^{AP'}$ of length $n+1$ given by $P_0 P_1 \dots P_n$, where (i) for all $k \in [1, n]$, $P_k = \{p_k\}$ if $p_k \in \sigma(i)$, and $P_k = \emptyset$ otherwise, and (ii) $P_0 = \{tag\}$ if $i > 0$ and $P_0 = \{tag, in\}$ otherwise. Note that proposition *in* marks the first position of $en(\sigma)$. Then, the finite Kripke structure $\mathcal{K}_{AP} = \langle S, S_0, E, Lab \rangle$ over AP' has size linear in $|AP|$ and it is constructed in such a way that its set of traces $\mathcal{L}(\mathcal{K}_{AP})$ is the set of the encodings of the traces over AP . Formally, \mathcal{K}_{AP} is defined as follows:

- $S = \{p_h, \bar{p}_h \mid h \in \{1, \dots, n\}\} \cup \{tag, in\}$ and $S_0 = \{in\}$;
- E consists of the edges (p_k, p_{k+1}) , (p_k, \bar{p}_{k+1}) , (\bar{p}_k, p_{k+1}) and $(\bar{p}_k, \bar{p}_{k+1})$ for all $k \in [1, n-1]$, and the edges (s, p_1) , (s, \bar{p}_1) , (p_n, tag) , and (\bar{p}_n, tag) where $s \in \{tag, in\}$.
- $Lab(tag) = \{tag\}$, $Lab(in) = \{in, tag\}$, and $Lab(p_k) = \{p_k\}$ and $Lab(\bar{p}_k) = \emptyset$ for all $k \in [1, n]$.

Let Λ be the set of pairs (h, ψ) consisting of a natural number $h \in [0, n]$ and a well-named QPTL formula ψ over AP such that there is no quantifier in ψ binding proposition p_h if $h \neq 0$, and ψ is a sentence iff $h = 0$. We inductively define a mapping assigning to each pair $(h, \psi) \in \Lambda$ a singleton-free SHyperLTL $_{S+C}^{\emptyset}$ formula $\mathbb{T}(h, \psi)$ over AP' and $\text{VAR} = \{x_1, \dots, x_n\}$ (intuitively, if $h \neq 0$, then p_h represents the currently quantified proposition):

- $\mathbb{T}(h, \top) = \top$;
- $\mathbb{T}(h, p_i) = \mathbf{X}^i p_i[x_h]$ for all $p_i \in AP$;
- $\mathbb{T}(h, \neg\psi) = \neg\mathbb{T}(h, \psi)$;
- $\mathbb{T}(h, \psi_1 \vee \psi_2) = \mathbb{T}(h, \psi_1) \vee \mathbb{T}(h, \psi_2)$;
- $\mathbb{T}(h, \mathbf{X}\psi) = \mathbf{X}^{n+1}\mathbb{T}(h, \psi)$;
- $\mathbb{T}(h, \mathbf{Y}\psi) = \mathbf{Y}^{n+1}\mathbb{T}(h, \psi)$;
- $\mathbb{T}(h, \psi_1 \mathbf{U} \psi_2) = (tag[x_h] \rightarrow \mathbb{T}(h, \psi_1)) \mathbf{U} (\mathbb{T}(h, \psi_2) \wedge tag[x_h])$;
- $\mathbb{T}(h, \psi_1 \mathbf{S} \psi_2) = (tag[x_h] \rightarrow \mathbb{T}(h, \psi_1)) \mathbf{S} (\mathbb{T}(h, \psi_2) \wedge tag[x_h])$;
- $$\mathbb{T}(h, \exists p_k.\psi) = \begin{cases} \exists^P x_k. \left(\mathbb{T}(k, \psi) \wedge \mathbf{O}(in[x_h] \wedge in[x_k]) \wedge \right. \\ \qquad \qquad \qquad \left. \mathbf{G} \bigwedge_{j \in [1, n] \setminus \{k\}} (p_j[x_h] \leftrightarrow p_j[x_k]) \right) & \text{if } h \neq 0 \\ \exists x_k. \mathbb{T}(k, \psi) & \text{otherwise} \end{cases}$$

By construction, $\mathbb{T}(h, \psi)$ has size linear in ψ and has the same strong alternation depth as ψ . Moreover, $\mathbb{T}(h, \psi)$ is a SHyperLTL $_{S+C}^{\emptyset}$ sentence iff ψ is a QPTL sentence. Now we show that the construction is correct. A $\mathcal{L}(\mathcal{K}_{AP})$ -assignment is a mapping $\Pi : \text{VAR} \rightarrow \mathcal{L}(\mathcal{K}_{AP})$. For all $i \geq 0$, $\mathcal{L}(\mathcal{K}_{AP})$ -assignments Π , and SHyperLTL $_{S+C}^{\emptyset}$ formulas φ over AP' , we write $(\Pi, i) \models \varphi$ to mean that $(\Pi_i, \text{VAR}) \models_{\mathcal{L}(\mathcal{K}_{AP})} \varphi$, where Π_i is the (pointed) trace assignment over $\mathcal{L}(\mathcal{K}_{AP})$ assigning to each trace variable $x \in \text{VAR}$, the pointed trace $(\Pi(x), i)$. Note that for each QPTL sentence ψ over AP , ψ is satisfiable if for all traces σ , $(\sigma, 0) \models \psi$. Thus, correctness of the construction directly follows from the following claim, where for each $i \geq 0$,

$\wp(i) := i \cdot (n + 1)$. Intuitively, $\wp(i)$ is the *tag*-position associated with the AP'-encoding of the position i of a trace over AP.

Claim. Let $(h, \psi) \in \Lambda$. Then, for all pointed traces (σ, i) over AP and $\mathcal{L}(\mathcal{K}_{\text{AP}})$ -assignments Π such that $\Pi(x_h) = \text{en}(\sigma)$ if $h \neq 0$, and $i = 0$ if $h = 0$, the following holds:

$$(\sigma, i) \models \psi \Leftrightarrow (\Pi, \wp(i)) \models \mathsf{T}(h, \psi)$$

The claim is proved by structural induction on ψ . The cases for the boolean connectives easily follow from the induction hypothesis. For the other cases, we proceed as follows:

- $\psi = p_j$ for some $p_j \in \text{AP}$: in this case $h \neq 0$ (recall that $(h, \psi) \in \Lambda$). Then, we have that $(\sigma, i) \models p_j \Leftrightarrow p_j \in \sigma(i) \Leftrightarrow p_j \in \text{en}(\sigma)(\wp(i) + j) \Leftrightarrow p_j \in \Pi(x_h)(\wp(i) + j) \Leftrightarrow (\Pi, \wp(i)) \models \mathbf{X}^j p_j[x_h] \Leftrightarrow (\Pi, \wp(i)) \models \mathsf{T}(h, p_j)$. Hence, the result follows.
- $\psi = \mathbf{X}\psi'$: hence, $h \neq 0$. We have that $(\sigma, i) \models \mathbf{X}\psi' \Leftrightarrow (\sigma, i + 1) \models \psi' \Leftrightarrow$ (by the induction hypothesis) $(\Pi, \wp(i + 1)) \models \mathsf{T}(h, \psi') \Leftrightarrow$ (since $\wp(i + 1) = \wp(i) + n + 1$) $(\Pi, \wp(i)) \models \mathbf{X}^{n+1}\mathsf{T}(h, \psi') \Leftrightarrow (\Pi, \wp(i)) \models \mathsf{T}(h, \mathbf{X}\psi')$. Hence, the result follows.
- $\psi = \mathbf{Y}\psi'$: similar to the previous case.
- $\psi = \psi_1 \mathbf{U} \psi_2$: hence, $h \neq 0$. We have that $(\sigma, i) \models \psi_1 \mathbf{U} \psi_2 \Leftrightarrow$ there is $j \geq i$ such that $(\sigma, j) \models \psi_2$ and $(\sigma, \ell) \models \psi_1$ for all $i \leq \ell < j \Leftrightarrow$ (by the induction hypothesis) there is $j \geq i$ such that $(\Pi, \wp(j)) \models \mathsf{T}(h, \psi_2)$ and $(\Pi, \wp(\ell)) \models \mathsf{T}(h, \psi_1)$ for all $i \leq \ell < j \Leftrightarrow$ there is $j' \geq \wp(i)$ such that $(\Pi, j') \models \mathsf{T}(h, \psi_2)$ and $\text{tag} \in \Pi(x_h)(j')$, and for all $\wp(i) \leq \ell' < j'$ such that $\text{tag} \in \Pi(x_h)(\ell')$, $(\Pi, \ell') \models \mathsf{T}(h, \psi_1) \Leftrightarrow (\Pi, \wp(i)) \models \mathsf{T}(h, \psi_1 \mathbf{U} \psi_2)$. Hence, the result follows.
- $\psi = \psi_1 \mathbf{S} \psi_2$: similar to the previous case.
- $\psi = \exists p_k. \psi'$ and $h \neq 0$: by hypothesis, $k \neq h$. For the implication, $(\Pi, \wp(i)) \models \mathsf{T}(h, \psi) \Rightarrow (\sigma, i) \models \psi$, assume that $(\Pi, \wp(i)) \models \mathsf{T}(h, \psi)$. By definition of $\mathsf{T}(h, \exists p_k. \psi')$ and since *in* marks the first position of the encoding of a trace over AP, it easily follows that there exists a pointed trace over AP of the form (σ', i) such that $\sigma' =_{\text{AP} \setminus \{p_k\}} \sigma$ and $(\Pi[x_k \leftarrow \text{en}(\sigma')], \wp(i)) \models \mathsf{T}(k, \psi')$. Since $(k, \psi') \in \Lambda$, by the induction hypothesis, it follows that $(\sigma', i) \models \psi'$. Thus, being $\sigma' =_{\text{AP} \setminus \{p_k\}} \sigma$, we obtain that $(\sigma, i) \models \psi$. The converse implication $(\sigma, i) \models \psi \Rightarrow (\Pi, \wp(i)) \models \mathsf{T}(h, \psi)$ is similar, and we omit the details here.
- $\psi = \exists p_k. \psi'$ and $h = 0$: hence, ψ and $\mathsf{T}(0, \psi)$ are sentences. We have that $(\sigma, 0) \models \exists p_k. \psi' \Leftrightarrow (\psi \text{ is a QPTL sentence})$ for some trace σ' over AP, $(\sigma', 0) \models \psi' \Leftrightarrow$ (by the induction hypothesis) $(\Pi[x_k \rightarrow \text{en}(\sigma')], \wp(0)) \models \mathsf{T}(k, \psi') \Leftrightarrow$ (by definition of $\mathsf{T}(0, \exists p_k. \psi')$) $(\Pi, \wp(0)) \models \mathsf{T}(0, \exists p_k. \psi')$.

This concludes the proof of Theorem 4.3. \blacktriangleleft

B.4 Proof of Proposition 4.5

In order to prove Proposition 4.5, we need some preliminary results. Recall that a Non-deterministic Büchi Automaton over words (NBA for short) is a tuple $\mathcal{A} = \langle \Sigma, Q, Q_0, \Delta, \text{Acc} \rangle$, where Σ is a finite alphabet, Q is a finite set of states, $Q_0 \subseteq Q$ is the set of initial states, $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation, and $\text{Acc} \subseteq Q$ is the set of *accepting* states. Given an infinite word w over Σ , a run of \mathcal{A} over w is an infinite sequence of states q_0, q_1, \dots such that $q_0 \in Q_0$ and for all $i \geq 0$, $(q_i, w(i), q_{i+1}) \in \Delta$. The run is accepting if for infinitely many i , $q_i \in \text{Acc}$. The language $\mathcal{L}(\mathcal{A})$ accepted by \mathcal{A} consists of the infinite words w over Σ such that there is an accepting run over w .

Fix a non-empty set Γ of PLTL formulas over AP. The closure $cl(\Gamma)$ of Γ is the set of PLTL formulas consisting of the formulas \top , $\mathbf{Y}\top$, the sub-formulas of the formulas $\theta \in \Gamma$,

and the negations of such formulas (we identify $\neg\neg\theta$ with θ). Note that $\Gamma \subseteq cl(\Gamma)$. Without loss of generality, we can assume that $AP \subseteq \Gamma$. Precisely, AP can be taken as the set of propositions occurring in the given simple GHyperLTL_{S+C} sentence and $cl(\Gamma)$ contains all the propositions in AP and their negations. For each formula $\theta \in cl(\Gamma) \setminus AP$, we introduce a fresh atomic proposition not in AP , denoted by $at(\theta)$. Moreover, for allowing a uniform notation, for each $p \in AP$, we write $at(p)$ to mean p itself. Let AP_Γ be the set AP extended with these new propositions. By a straightforward adaptation of the well-known translation of PLTL formulas into equivalent NBA [37], we obtain the following result, where for a trace σ_Γ over AP_Γ , $(\sigma_\Gamma)_{AP}$ denotes the projection of σ_Γ over AP .

- **Proposition B.6.** *Given a finite set Γ of PLTL formulas over AP , one can construct in single exponential time an NBA \mathcal{A}_Γ over 2^{AP_Γ} with $2^{O(|AP_\Gamma|)}$ states satisfying the following:*
1. *let $\sigma_\Gamma \in \mathcal{L}(\mathcal{A}_\Gamma)$: then for all $i \geq 0$ and $\theta \in cl(\Gamma)$, $at(\theta) \in \sigma_\Gamma(i)$ iff $((\sigma_\Gamma)_{AP}, i) \models \theta$.*
 2. *for each trace σ over AP , there exists $\sigma_\Gamma \in \mathcal{L}(\mathcal{A}_\Gamma)$ such that $\sigma = (\sigma_\Gamma)_{AP}$.*

Proof. Here, we construct a *generalized* NBA \mathcal{A}_Γ satisfying Properties (1) and (2) of Proposition B.6, which can be converted in linear time into an equivalent NBA. Recall that a generalized NBA is defined as an NBA but the acceptance condition is given by a family $\mathcal{F} = \{Acc_1, \dots, Acc_k\}$ of sets of accepting states. In this case, a run is accepting if for each accepting component $Acc_i \in \mathcal{F}$, the run visits infinitely often states in Acc_i .

The generalized NBA $\mathcal{A}_\Gamma = \langle 2^{AP_\Gamma}, Q, Q_0, \Delta, \mathcal{F} \rangle$ is defined as follows. Q is the set of *atoms* of Γ consisting of the maximal propositionally consistent subsets A of $cl(\Gamma)$. Formally, an atom A of Γ is a subset of $cl(\Gamma)$ satisfying the following:

- $\top \in A$
- for each $\theta \in cl(\Gamma)$, $\theta \in A$ iff $\neg\theta \notin A$;
- for each $\theta_1 \vee \theta_2 \in cl(\Gamma)$, $\theta_1 \vee \theta_2 \in A$ iff $\{\theta_1, \theta_2\} \cap A \neq \emptyset$.

The set Q_0 of initial states consists of the atoms A of Γ such that $\neg\mathbf{Y}\top \in A$. For an atom A , $at(A)$ denotes the subset of propositions in AP_Γ associated with the formulas in A , i.e. $at(A) := \{at(\theta) \mid \theta \in A\}$. The transition relation Δ captures the semantics of the next and previous modalities and the local fixpoint characterization of the until and since modalities. Formally, Δ consists of the transitions of the form $(A, at(A), A')$ such that:

- for each $\mathbf{X}\theta \in cl(\Gamma)$, $\mathbf{X}\theta \in A$ iff $\theta \in A'$;
- for each $\mathbf{Y}\theta \in cl(\Gamma)$, $\mathbf{Y}\theta \in A'$ iff $\theta \in A$;
- for each $\theta_1 \mathbf{U} \theta_2 \in cl(\Gamma)$, $\theta_1 \mathbf{U} \theta_2 \in A$ iff either $\theta_2 \in A$, or $\theta_1 \in A$ and $\theta_1 \mathbf{U} \theta_2 \in A'$;
- for each $\theta_1 \mathbf{S} \theta_2 \in cl(\Gamma)$, $\theta_1 \mathbf{S} \theta_2 \in A'$ iff either $\theta_2 \in A'$, or $\theta_1 \in A'$ and $\theta_1 \mathbf{S} \theta_2 \in A$.

Finally, the generalized Büchi acceptance condition is used for ensuring the fulfillment of the liveness requirements θ_2 in the until sub-formulas $\theta_1 \mathbf{U} \theta_2$ in Γ . Formally, for each $\theta_1 \mathbf{U} \theta_2 \in cl(\Gamma)$, \mathcal{F} has a component consisting of the atoms A such that either $\neg(\theta_1 \mathbf{U} \theta_2) \in A$ or $\theta_2 \in A$.

Let $\sigma_\Gamma \in \mathcal{L}(\mathcal{A}_\Gamma)$. By construction, there is an accepting infinite sequence of atoms $\rho = A_0 A_1 \dots$ such that for all $i \geq 0$, $\sigma_\Gamma(i) = at(A_i)$. Let σ be the projection of σ_Γ over AP (note that $A_i \cap AP = \sigma(i)$ for all $i \geq 0$). By standard arguments (see [37]), the following holds: for all $i \geq 0$ and $\theta \in cl(\Gamma)$, $\theta \in A_i$ (hence, $at(\theta) \in \sigma_\Gamma(i)$) if and only if $(\sigma, i) \models \theta$. Hence, Property (1) of Proposition B.6 follows.

For Property (2), let σ be a trace over AP and let $\rho = A_0 A_1 \dots$ be the infinite sequence of atoms defined as follows for all $i \geq 0$: $A_i = \{\theta \in cl(\Gamma) \mid (\sigma, i) \models \theta\}$. By construction and the semantics of PLTL, ρ is an accepting run of \mathcal{A}_Γ over the word $\sigma_\Gamma = at(A_0)at(A_1)\dots$. Moreover, σ coincides with the projection of σ_Γ over AP . Hence, the result follows. ◀

Let $\mathcal{K} = \langle S, S_0, E, Lab \rangle$ be a finite Kripke structure over AP and $F \subseteq S$. Next, we consider the synchronous product of the fair Kripke structure (\mathcal{K}, F) with the NBA $\mathcal{A}_\Gamma = \langle 2^{AP_\Gamma}, Q, Q_0, \Delta, Acc \rangle$ over 2^{AP_Γ} of Proposition B.6 associated with Γ . More specifically, we construct a Kripke structure \mathcal{K}_Γ over AP_Γ and a subset F_Γ of \mathcal{K}_Γ -states such that $\mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)$ is the set of traces $\sigma_\Gamma \in \mathcal{L}(\mathcal{A}_\Gamma)$ whose projections over AP are in $\mathcal{L}(\mathcal{K}, F)$. Formally, the Γ -extension of (\mathcal{K}, F) is the fair Kripke structure $(\mathcal{K}_\Gamma, F_\Gamma)$ where $\mathcal{K}_\Gamma = \langle S_\Gamma, S_{0,\Gamma}, E_\Gamma, Lab_\Gamma \rangle$ and F_Γ are defined as follows:

- S_Γ is the set of tuples $(s, B, q, \ell) \in S \times 2^{AP_\Gamma} \times Q \times \{1, 2\}$ such that $Lab(s) = B \cap AP$;
- $S_{0,\Gamma} = S_\Gamma \cap (S_0 \times 2^{AP_\Gamma} \times Q_0 \times \{1\})$;
- E_Γ consists of the following transitions:
 - $((s, B, q, 1), (s', B', q', \ell))$ such that $(s, s') \in E$, $(q, B, q') \in \Delta$, and $\ell = 2$ if $s \in F$ and $\ell = 1$ otherwise;
 - $((s, B, q, 2), (s', B', q', \ell))$ such that $(s, s') \in E$, $(q, B, q') \in \Delta$, and $\ell = 1$ if $q \in Acc$ and $\ell = 2$ otherwise.
- for each $(s, B, q, \ell) \in S_\Gamma$, $Lab_\Gamma((s, B, q, \ell)) = B$;
- $F_\Gamma = \{(s, B, q, 2) \in S_\Gamma \mid q \in Acc\}$.

By construction and Proposition B.6(2), we easily obtain the following result.

► **Proposition B.7.** *For each trace σ_Γ over AP_Γ , $\sigma_\Gamma \in \mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)$ if and only if $\sigma_\Gamma \in \mathcal{L}(\mathcal{A}_\Gamma)$ and $(\sigma_\Gamma)_{AP} \in \mathcal{L}(\mathcal{K}, F)$. Moreover, for each $\sigma \in \mathcal{L}(\mathcal{K}, F)$, there exists $\sigma_\Gamma \in \mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)$ such that $(\sigma_\Gamma)_{AP} = \sigma$.*

We can now provide a proof of Proposition 4.5.

► **Proposition 4.5.** *Given a simple GHyperLTL_{S+C} sentence φ and a fair finite Kripke structure (\mathcal{K}, F) over AP , one can build in single exponential time in the size of φ , a fair finite Kripke structure (\mathcal{K}', F') over an extension AP' of AP and a singleton-free $\text{SHyperLTL}_{S+C}^{\Gamma'}$ sentence φ' for some $\Gamma' \subseteq AP'$ such that $\mathcal{L}(\mathcal{K}', F') \models \varphi'$ if and only if $\mathcal{L}(\mathcal{K}, F) \models \varphi$. Moreover, φ' has the same strong alternation depth as φ , $|\varphi'| = O(|\varphi|)$, and $|\mathcal{K}'| = O(|\mathcal{K}| * 2^{O(|\varphi|)})$.*

Proof. Let φ be a GHyperLTL_{S+C} sentence over AP and (\mathcal{K}, F) be a fair finite Kripke structure (\mathcal{K}, F) over AP . Then, there is a finite set Γ_0 of PLTL formulas such that φ is in the fragment $\text{SHyperLTL}_{S+C}^{\Gamma_0}$. Let Γ be the set of PLTL formulas consisting of the formulas in Γ_0 and the PLTL formulas ψ such that $\langle x \rangle \psi[x]$ is a sub-formula of φ for some variable x . Define AP' , $\Gamma' \subseteq AP'$, (\mathcal{K}', F') , and φ' as follows:

- $AP' := AP_\Gamma$ (recall that $AP_\Gamma = AP \cup \{at(\theta) \mid \theta \in \Gamma\}$) and Γ' is the AP_Γ -counterpart of Γ_0 , i.e. $\Gamma' := \{at(\theta) \mid \theta \in \Gamma_0\}$;
- $(\mathcal{K}', F') := (\mathcal{K}_\Gamma, F_\Gamma)$, where $(\mathcal{K}_\Gamma, F_\Gamma)$ is the Γ -extension of (\mathcal{K}, F) ;
- $\varphi' := \mathbb{T}(\varphi)$ where the mapping \mathbb{T} replaces (i) each sub-formula $\langle x \rangle \psi[x]$ of φ with its propositional x -version $at(\psi)[x]$, and (ii) each Γ_0 -relativized temporal modality with its Γ' -relativized version. Note that $\mathbb{T}(\varphi)$ is a singleton-free $\text{SHyperLTL}_{S+C}^{\Gamma'}$ formula where Γ' is propositional (in particular, $\Gamma' \subseteq AP'$) and has the same strong alternation depth as φ .

It remains to show that the construction is correct, i.e. $\mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma) \models \mathbb{T}(\varphi)$ iff $\mathcal{L}(\mathcal{K}, F) \models \varphi$. Let Λ be the set of formulas ϕ in the fragment $\text{SHyperLTL}_{S+C}^{\Gamma_0}$ such that for each sub-formula $\langle x \rangle \psi[x]$ of ϕ , it holds that $\psi \in \Gamma$. Moreover, for a trace assignment Π_Γ over $\mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)$, the AP -projection of Π_Γ , written $(\Pi_\Gamma)_{AP}$, is the trace assignment with domain $Dom(\Pi)$ obtained from Π by replacing each pointed trace $\Pi(x)$, where $x \in Dom(\Pi)$ and $\Pi(x)$ is of the form (σ_Γ, i) , with $((\sigma_\Gamma)_{AP}, i)$. Note that by Proposition B.7, $(\Pi_\Gamma)_{AP}$ is a trace assignment over $\mathcal{L}(\mathcal{K}, F)$. Correctness of the construction directly follows from the following claim.

15:36 Unifying Asynchronous Logics for Hyperproperties

Claim. Let $\phi \in \Lambda$ and Π_Γ be a trace assignment over $\mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)$. Then:

$$(\Pi_\Gamma, \text{VAR}) \models_{\mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)} \top(\phi) \text{ iff } ((\Pi_\Gamma)_{\text{AP}}, \text{VAR}) \models_{\mathcal{L}(\mathcal{K}, F)} \phi$$

The claim is proved by structural induction on $\phi \in \Lambda$. The cases where the root modality of ϕ is a Boolean connective directly follow from the induction hypothesis. For the other cases, we proceed as follows.

- $\phi = \langle x \rangle \psi[x]$, where $\psi \in \Gamma$. Hence, $\top(\phi) = \text{at}(\psi)[x]$. Let $\Pi_\Gamma(x) = (\sigma_\Gamma, i)$. We have that $(\Pi_\Gamma)_{\text{AP}}(x) = ((\sigma_\Gamma)_{\text{AP}}, i)$. By Propositions B.6 and B.7, $\text{at}(\psi) \in \sigma_\Gamma(i)$ iff $((\sigma_\Gamma)_{\text{AP}}, i) \models \psi$. Hence, the result follows.
- The root modality of ϕ is a Γ_0 -relativized temporal modality. Assume that $\phi = \phi_1 \mathbf{U}_{\Gamma_0} \phi_2$ (the other cases being similar). Then, $\top(\phi) = \top(\phi_1) \mathbf{U}_{\Gamma'} \top(\phi_2)$ (recall that $\Gamma' = \{\text{at}(\theta) \mid \theta \in \Gamma_0\}$ and $\Gamma_0 \subseteq \Gamma$). By Propositions B.6 and B.7, for each pointed trace (σ_Γ, i) over $\mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)$ and $\theta \in \Gamma_0$, it holds that $\text{at}(\theta) \in \sigma_\Gamma(i)$ iff $((\sigma_\Gamma)_{\text{AP}}, i) \models \theta$. By construction, it follows that $\text{succ}_{(\Gamma_0, \text{VAR})}((\Pi_\Gamma)_{\text{AP}})$ is the AP-projection of $\text{succ}_{(\Gamma, \text{VAR})}(\Pi_\Gamma)$. Hence, by the semantics of GHyperLTL_{S+C} and the induction hypothesis, the result follows.
- $\phi = \exists x. \phi'$. Hence, $\top(\phi) = \exists x. \top(\phi')$. By Proposition B.7, for each trace $\sigma \in \mathcal{L}(\mathcal{K}, F)$, there exists $\sigma_\Gamma \in \mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)$ such that $(\sigma_\Gamma)_{\text{AP}} = \sigma$. Hence, the result easily follows from the induction hypothesis.
- $\phi = \exists^P x. \phi'$: this case is similar to the previous one.

◀

B.5 Proof of Proposition 4.7

► **Proposition 4.7.** *Given $\emptyset \neq \Gamma \subseteq \text{AP}$ and a fair finite Kripke structure (\mathcal{K}, F) over AP, one can construct in polynomial time a fair finite Kripke structure $(\mathcal{K}_\Gamma, F_\Gamma)$ and a LTL formula θ_Γ such that $\text{stfr}_\Gamma^\#(\mathcal{L}(\mathcal{K}, F))$ is the set of traces $\sigma \in \mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)$ so that $\sigma \models \theta_\Gamma$.*

Proof. Let $\mathcal{K} = \langle S, S_0, E, \text{Lab} \rangle$. Intuitively, the Kripke structure \mathcal{K}_Γ is obtained from \mathcal{K} by adding edges which are *summaries* of finite paths π of \mathcal{K} where *either* the propositional valuation in Γ changes only at the final state of π , or the propositional valuation in Γ does not change along π . Formally, let $R_\Gamma(\mathcal{K})$ and $R_\Gamma(\mathcal{K}, F)$ be the sets of state pairs in \mathcal{K} defined as follows:

- $R_\Gamma(\mathcal{K})$ consists of the pairs $(s, s') \in S \times S$ such that $\text{Lab}(s) \cap \Gamma \neq \text{Lab}(s') \cap \Gamma$ and there is a finite path of \mathcal{K} of the form $s \cdot \rho \cdot s'$ such that $\text{Lab}(s) \cap \Gamma = \text{Lab}(\rho(i)) \cap \Gamma$ for all $0 \leq i < |\rho|$.
- $R_\Gamma(\mathcal{K}, F)$ is defined similarly but, additionally, we require that the finite path $s \cdot \rho \cdot s'$ visits some state in F .

Intuitively, $R_\Gamma(\mathcal{K})$ keeps track of the initial and final states of the finite paths of \mathcal{K} with length at least 2 where the propositional valuation in Γ changes only at the final state of the finite path. Additionally, $R_\Gamma(\mathcal{K}, F)$ considers only those finite paths which visit some state in F . Moreover, let $R_\Gamma^\#(\mathcal{K})$ and $R_\Gamma^\#(\mathcal{K}, F)$ be the sets of state pairs in \mathcal{K} defined as follows:

- $R_\Gamma^\#(\mathcal{K})$ consists of the pairs $(s, s') \in S \times S$ such that $\text{Lab}(s) \cap \Gamma = \text{Lab}(s') \cap \Gamma$ and there is a finite path of \mathcal{K} of the form $s \cdot \rho \cdot s'$ such that $\text{Lab}(s) \cap \Gamma = \text{Lab}(\rho(i)) \cap \Gamma$ for all $0 \leq i < |\rho|$.
- $E_\Gamma^\#(\mathcal{K}, F)$ is defined similarly but, additionally, we require that the finite path $s \cdot \rho \cdot s'$ visits some accepting state in F .

Thus, $R_\Gamma^\#(\mathcal{K})$ keeps track of the initial and final states of the finite paths of \mathcal{K} with length at least 2 where the propositional valuation in Γ does not change. Additionally, $R_\Gamma^\#(\mathcal{K}, F)$ considers only those finite paths which visit some state in F . The finite sets $R_\Gamma(\mathcal{K})$, $R_\Gamma(\mathcal{K}, F)$,

$R_\Gamma^\#(\mathcal{K})$, $R_\Gamma^\#(\mathcal{K}, F)$ can be easily computed in polynomial time by standard closure algorithms. By exploiting these finite sets, we define the finite Kripke structure $\mathcal{K}_\Gamma = \langle S_\Gamma, S_{\Gamma,0}, E_\Gamma, Lab_\Gamma \rangle$ and the set $F_\Gamma \subseteq S_\Gamma$ as follows.

- S_Γ is given by $S \times 2^{\{acc, \#\}}$ and $S_{\Gamma,0}$ is the set of states of the form (s, \emptyset) for some $s \in S_0$.
- E_Γ consists of the edges $((s, T), (s', T'))$ such that one of the following holds:
 - $(s, s') \in E \cup R_\Gamma(\mathcal{K})$, $\# \notin T'$, and $(acc \in T' \text{ iff } s' \in F)$;
 - $(s, s') \in R_\Gamma(\mathcal{K}, F)$, $\# \notin T'$, and $acc \in T'$;
 - $(s, s') \in R_\Gamma^\#(\mathcal{K})$, $\# \notin T$, $\# \in T'$, and $(acc \in T' \text{ iff } s' \in F)$;
 - $(s, s') \in R_\Gamma^\#(\mathcal{K}, F)$, $\# \notin T$, $\# \in T'$, and $acc \in T'$.
- $Lab_\Gamma(s, T) = Lab(s) \cup \{\#\}$ if $\# \in T$, and $Lab_\Gamma(s, T) = Lab(s)$ otherwise;
- F_Γ is the set of \mathcal{K}_Γ -states (s, T) such that $acc \in T$.

Intuitively, proposition $\#$ marks only the \mathcal{K} -states which are targets of pairs in $R_\Gamma^\#(\mathcal{K}) \cup R_\Gamma^\#(\mathcal{K}, F)$, while the flag acc marks either the states in F which are targets of \mathcal{K} -edges, or the \mathcal{K} -states which are targets of pairs in $R_\Gamma(\mathcal{K}, F) \cup R_\Gamma^\#(\mathcal{K}, F)$. We say that a trace σ over $AP \cup \{\#\}$ is *well-formed* if one of the following conditions holds:

- or σ is a Γ -stutter free trace over AP (i.e. $stfr_\Gamma(\sigma) = \sigma$);
- or there is a position $i \geq 0$ such that $i + 1$ is the unique position where $\#$ holds and $stfr_\Gamma(\sigma) = \sigma(0) \dots \sigma(i) \cdot \sigma^{i+2}$.

By construction, it easily follows that $stfr_\Gamma^\#(\mathcal{L}(\mathcal{K}, F))$ is the set of *well-formed* traces in $\mathcal{L}(\mathcal{K}_\Gamma, F_\Gamma)$. Then, the LTL formula θ_Γ captures the well-formed requirement and is defined as follows.

$$\neg\# \wedge \mathbf{G}(\# \rightarrow \mathbf{X} \mathbf{G} \neg\#) \wedge \mathbf{G}\left(\bigvee_{p \in \Gamma} ((p \leftrightarrow \neg \mathbf{X}p) \wedge \neg\# \wedge \mathbf{X}\neg\#) \vee \bigwedge_{p \in \Gamma} \mathbf{G}((p \leftrightarrow \mathbf{X}p) \wedge \neg\#) \vee (\mathbf{X}\# \wedge \bigwedge_{p \in \Gamma} (p \leftrightarrow \mathbf{X}p) \wedge \bigvee_{p \in \Gamma} (p \leftrightarrow \neg \mathbf{X}^2 \neg p))\right)$$

This concludes the proof of Proposition 4.7. ◀